Foundations of neo-Bayesian statistics

Spinu Vitalie

April 9, 2009
Outline

1. What is Uncertainty?
   - A Definition
   - Probability & Statistics
   - Alternatives for Probability Calculus

2. Examples
   - Examples: Information Representation
   - Examples: Behavioral Violations of EU

3. Behavioral Foundations: Alternatives to EU
Outline

1 What is Uncertainty?
   • A Definition
     • Probability & Statistics
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2 Examples
   • Examples: Information Representation
   • Examples: Behavioral Violations of EU

3 Behavioral Foundations: Alternatives to EU
What is Uncertainty?
Or why the life at TI is so certain?

- Which TI seminar to go to?
- What journal to choose for your publication? Will your paper be accepted or not?
- Will you get stuck in elevator?
- Will the gym be full or not?
- Take raincoat or not?
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What is Uncertainty?

Uncertain (at Wiktionary.org)
Not known for certain; questionable; not yet determined; undecided; variable and subject to change; fitful or unsteady

- **Aleatory Uncertainty**
  - Casino gambling; Radioactive decay rate; Random Number Generators; Urns with known number of colored balls etc.
  - Sampled from $p(\bullet)$

- **Mixed Uncertainty**
  - Data are sampled from $p(\bullet|\theta)$ when $\theta$ is unknown

- **Epistemic Uncertainty**
  - Queen Beatrix’s birthday; Your weight; Population of Netherlands in 2008 etc.
  - Queen Beatrix’s death day; Population of Netherlands in 2020 etc.
  - One possible model: $p(\bullet|M_0)$, $\theta \in \Theta$
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Rain Example

- State space: $\Omega = \{\text{Heavy Rain, Light Rain, No Rain}\}$
- Multinomial: $x \sim f(p_H, p_L, p_N, 1)$
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Probability vs Statistics

- Probability is undisputed model for DGP!

- What is inferential statistics?

- How to formalize the inversion problem from above?
  - Frequentist approach (LLN, CLT, Kolmogorov theorem)
    - Method of Moments (substitution principle)
    - Maximum Likelihood (ML)
  - Decision Theoretic approach
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    - Risk Functions (admissibility, minimaxity)
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Probability vs Statistics

- Probability is undisputed model for DGP! $\theta \xrightarrow{p(x|\theta)} X$
- What is inferential statistics? $\hat{\theta} \xleftarrow{l(\theta|x)} X$
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Decision Theoretic Framework

Inversion problem

Data is sampled from $p(\bullet|\theta)$ when $\theta$ is unknown

\[ \theta \xrightarrow{p(x|\theta)} X \]
\[ \hat{\theta} \leftarrow X \]

\[ \min \int U(\hat{\theta} - \theta)p(\theta|x)d\theta \]

$U$ is a loss function and some times is denoted as $L$. 

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$p(X|\theta)\pi(\theta)$

$aU$ is a loss function and sometimes is denoted as $L$
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Behavioral Foundations: Alternatives to EU

A Definition

Probability & Statistics

Alternatives for Probability Calculus

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3. convex monotonic measures (Choquet capacities of order 2) [G. Choquet(1954), D. Denneberg(1994)],


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A Definition

Probability & Statistics

Alternatives for Probability Calculus

Probability Triangle

\[ \Omega = \{ \omega_1, \omega_2, \omega_3 \} \]

\[ p = ( p(\omega_1), p(\omega_2), p(\omega_3) ) \]

\[ p \in A \]

\[ x = ( x(\omega_1), x(\omega_2), x(\omega_3) ) \]

\[ P(x) = \min_{(p_1, p_2, p_3) \in A} (x_1p_1 + x_2p_2 + x_3p_3) \]
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Probability Triangle

- $\Omega = \{\omega_1, \omega_2, \omega_3\}$
- $p = (p(\omega_1), p(\omega_2), p(\omega_3))$
- $p \in A$
- $x = (x(\omega_1), x(\omega_2), x(\omega_3))$
- $P(x) = \min_{(p_1, p_2, p_3) \in A} (x_1 p_1 + x_2 p_2 + x_3 p_3)$
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Behavioral Foundations: Alternatives to EU

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- \( p = (p(\omega_1), p(\omega_2), p(\omega_3)) \)
- \( p \in A \)
- \( x = (x(\omega_1), x(\omega_2), x(\omega_3)) \)
- \( P(x) = \min_{(\rho_1, \rho_2, \rho_3) \in A} (x_1 \rho_1 + x_2 \rho_2 + x_3 \rho_3) \)
Probability Triangle

- $\Omega = \{\omega_1, \omega_2, \omega_3\}$
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Example: Uncertain Urn

\[ \Omega = \{ \bullet, \circ \} \]

- \( \frac{4}{10} \leq P(\bullet) \leq \frac{7}{10} \)
- \( \frac{3}{10} \leq P(\circ) \leq \frac{6}{10} \)
What is Uncertainty?

Examples:

Behavioral Foundations: Alternatives to EU

Example: Uncertain Urn

\[ \Omega = \{\text{red}, \text{green}\} \]

\[ \frac{4}{10} \leq P(\text{red}) \leq \frac{7}{10} \]

\[ \frac{3}{10} \leq P(\text{green}) \leq \frac{6}{10} \]
Example: Uncertain Urn

- \( \Omega = \{\red{\ast}, \green{\ast}\} \)
- \( \frac{4}{10} \leq P(\red{\ast}) \leq \frac{7}{10} \)
- \( \frac{3}{10} \leq P(\green{\ast}) \leq \frac{6}{10} \)
Example: Coins Toss

Suppose that a fair coin is ‘tossed’ twice, in such a way that heads and tails are equally likely on each of the tosses but there is unknown interaction or dependence between the outcomes.

- $\Omega = \{H_1, H_2, H_1T_2, T_1H_2, T_1T_2\}$
- Perfect positive relationship: $p(\bullet) = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$
- Perfect negative relationship: $p(\bullet) = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$
Example: Coins Toss

Suppose that a fair coin is ‘tossed’ twice, in such a way that heads and tails are equally likely on each of the tosses but there is unknown interaction or dependence between the outcomes.

\[ \Omega = \{H_1 H_2, H_1 T_2, T_1 H_2, T_1 T_2\} \]

- Perfect positive relationship: \( p(\bullet) = (\frac{1}{2}, 0, 0, \frac{1}{2}) \)
- Perfect negative relationship: \( p(\bullet) = (0, \frac{1}{2}, \frac{1}{2}, 0) \)

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<tr>
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<th>( H_2 )</th>
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<tr>
<td>( H_1 )</td>
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<td>( \frac{1}{2} )</td>
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<tr>
<td>$H_1$</td>
<td>$\alpha$</td>
<td>$\frac{1}{2} - \alpha$</td>
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<tr>
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$\alpha \in [0, \frac{1}{2}]$
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\[
\begin{array}{ccc}
& H_2 & T_2 \\
H_1 & \alpha & 1/2 - \alpha & \frac{1}{2} \\
T_1 & 1/2 - \alpha & \alpha & \frac{1}{2} \\
\end{array}
\]

\[ \alpha \in [0, \frac{1}{2}] \]
Consider a football game with three possible outcomes for the home team \( \Omega = \{W(\text{win}), D(\text{draw}), L(\text{loss})\} \)

1. **not win** is at least as probable as **win**
   \[ P(W) \geq \frac{1}{2} \]
2. **win** is at least as probable as **draw**
   \[ P(W) \geq P(D) \]
3. **draw** is at least as probable as **loss**
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   \[ P(D) \geq P(L) \]
Example: Imprecision of natural language

Mary is 'young'


\[ Pos(A) = \sup_{a \in A} Pos(a) \]
Example: Imprecision of natural language

Mary is 'young'


\[ \text{Pos}(A) = \sup_{a \in A} \text{Pos}(a) \]
Outline

1. What is Uncertainty?
   - A Definition
   - Probability & Statistics
   - Alternatives for Probability Calculus

2. Examples
   - Examples: Information Representation
   - Examples: Behavioral Violations of EU

3. Behavioral Foundations: Alternatives to EU
Example: Allais Paradox

Choice 1:

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<tr>
<td>A</td>
<td>0 mil €</td>
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0.02 \cdot U(0) + 0.1 \cdot U(5) > 0.02 \cdot U(1) + 0.1 \cdot U(1)
Example: Allais Paradox

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Example: Allais Paradox

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Example: Elsberg Urns
Example: Elsberg Urns
Behavioral Foundations

Axioms

1. **Transitivity** \( x \succeq y \succeq z \Rightarrow x \succeq z \)

2. **Continuity**

3. **Completeness**

4. **Independence**

\[
x \succeq y \Rightarrow \alpha x + (1 - \alpha) z \succeq \alpha y + (1 - \alpha) z
\]

**Expected Utility Model**

\[
\int U \left( \hat{\theta} - \theta \right) \ast p(\theta) \, d\theta
\]
Behavioral Foundations

Axioms

1. Transitivity: \( x \succeq y \succeq z \Rightarrow x \succeq z \)

2. Continuity

3. Independence

\[
\alpha x + (1 - \alpha) z \succeq \alpha y + (1 - \alpha) z
\]

Bewley (2001) Unanimity Priors Model

\[
\int U (\hat{\theta} - \theta) \ast p(\theta) \, d\theta, \quad p(\bullet) \in \mathbb{P}
\]
**Behavioral Foundations**

### Axioms

1. **Transitivity**
   
   \[ x \succeq y \succeq z \Rightarrow x \succeq z \]

2. **Continuity**

3. **Completeness**

4. **Certainty Independence + Uncertainty Aversion**

   \[
   x \succeq y \Rightarrow \alpha x + (1 - \alpha) h \succeq \alpha y + (1 - \alpha) h, \quad h - certain
   \]

   \[
   x \asymp y \Rightarrow \alpha x + (1 - \alpha) y \succeq x
   \]

---


\[
\min_{\mathbf{p}(\theta) \in \mathcal{P}} \int U(\hat{\theta} - \theta) * \mathbf{p}(\theta) \, d\theta
\]
Behavioral Foundations

Axioms

1. Transitivity \( x \succeq y \succeq z \Rightarrow x \succeq z \)
2. Continuity
3. Completeness
4. Independence on Commonotonic Sets + Uncertainty Aversion

\[ x \succeq y \Rightarrow \alpha x + (1 - \alpha) h \succeq \alpha y + (1 - \alpha) h, \quad x, y, h \in \text{same set} \]

\[ x \asymp y \Rightarrow \alpha x + (1 - \alpha) y \succeq x \]

Schmeidler (1986) Rank Dependent Utility

\[ \min_{p(\theta) \in \mathbb{P}} \int U(\hat{\theta} - \theta) \ast p(\theta) d\theta, \quad \mathbb{P} - \text{polyhedral} \]

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Conclusions

- "The fundamental difficulties with the Bayesian theory concern the dogma of precision." (Walley 1991)
- Probability cannot adequately model:
  - ignorance,
  - partial information,
  - assessments of uncertainty in natural language, or
  - conflict between expert opinions.
- Rationality assumptions of EU are too strong for descriptive purpose
- EU cannot be coupled with alternative probability theories to form a formal inference model.