From Simple to Complex: a general method for extending behavioral foundations to infinitely many outcomes

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Abstract

This paper introduces a general method for extending decision models from simple choice alternatives (prospects) to larger domains of interest. Simple objects are usually functions taking finitely many values, as with simple acts in decision under uncertainty, or simple lotteries in decision under risk. The whole set of objects of interest often includes continuous and sometimes unbounded prospects. The proposed procedure is agnostic to the type of objects or the form of the representation functional, and can serve as a general add-on for virtually all presently existing behavioral representations.

Applying our procedure, the most general representations are obtained of expected utility, rank-dependent utility, prospect theory, and Choquet expected utility available in the literature. This paper also extends Fishburn's (1983) betweenes model, Gul's (1991) disappointment aversion model, and Gilboa and Schmeidler's (1989) MaxMin expected utility, models that had not been extended beyond simple prospects before.

A model-free discussion of "paradoxes" that can occur on larger domains is also provided, including the St. Petersburg paradox and the infamous "boundedness of utility" result for expected utility.

1 Introduction

A measurement functional (or model), which quantifies some physical or psychological variables (Krantz et al., 1971), is often justified by a set of mathematical axioms on the underlying ordering generated by the functional.

These axioms state necessary and sufficient conditions for the measurement model to hold, and constitute an *axiomatic representation* of the measurement.

A sophisticated measurement functional may, at first sight, entail an obscure mathematical representation that is difficult to understand. Binary orderings, to the contrary, are easy to grasp and interpret. Once the properties of the functional are summarized by the properties of the underlying binary ordering, the functional becomes easier to interpret and accept as a normative or descriptive tool.

In behavioral sciences and decision theory, the binary order induced by a decision model is often called *preference*, and the axiomatic representation is known as *behavioral foundation*. Because a binary order is the result of directly observable individual choices, behavioral foundations state conditions directly in terms of observables. They show how to directly verify or falsify the model empirically, and how to defend or criticize it normatively. If the preference conditions are natural, then the soundness of the model has been established.

A common strategy in axiomatizations of a measurement functional is to establish its validity on a subset of simple and well understood objects, and then extend it to more complex objects of interest. Take surface area measurement as an example. It is natural to first establish a measurement strategy of the area of squares and their disjoint unions and, only then, approximate the area of an arbitrary geometric form. For expected utility (Savage, 1954), simple objects are all acts taking finitely many outcomes. For discounted utility (Koopmans, 1972), simple objects are income streams with only finitely many "prizes" over time. Probability distributions with finitely many outcomes are yet another example of simple objects.

This paper is concerned with the second stage of the aforementioned strategy. It provides an unifying framework for the extension of measurement representations the set of simple to the whole set of complex objects in a general and simple setup. Without using continuity or topological assumptions, the proposed framework is based solely on intuitive denseness conditions of the type, if $h \succ l$ then there must exist a special simple object l^s such that: $h \succ l^s \succcurlyeq l$. The main extension theorems (section 3) can be viewed as a general add-on to aid researchers in the process of extending preference representations. The researchers can simply focus on providing their theorems for simple objects, and then apply these extension theorems to obtain representations for their whole set of interest.

An axiomatization of a functional on the set of simple objects is domain and model specific. Under a general set of assumptions, the extension from simple object to complex ones can be abstracted from the specific form of the measurement functional, or the nature of the objects. This extension process can be further split into two sub-stages – extending from simple to bounded objects, and then extending from bounded to unbounded objects. Extending from simple to bounded is usually straightforward in the presence of monotonicity or continuity conditions. This is done by "sandwiching" complex objects between simple ones. Extending from bounded to unbounded is more demanding, as illustrated by the papers attempting to address this issue in general settings – Fishburn (1975), Wakker (1993), Kopylov (2011) and Kothiyal et al. (2011).

Providing a representation on an unwieldy large set of objects usually leads to undesirable restrictions, as with the well know case of bounded utility in EU representations ((Arrow, 1974; Bassett, 1987; Fishburn, 1967, 1975; Menger, 1934; Ryan, 1974)). If utility is unbounded and the set of prospects of interests is too large, then St. Petersburg prospects can be constructed, thus violating the definition of finite valued expected utility. Hence, utility must be bounded. Even when expected utility is allowed to take infinite values, then it is possible to find two prospects a and b such that a is strictly better than b, but the expected utility of both is infinite, a contradiction. To avoid such undesired consequences, the set of objects should be restricted in one way or another. The approach taken in this paper achieves maximal generality in this respect, by imposing minimal restrictions on the underlying set. As a consequence of the adopted setup, any object with finite value of the measurement functional can be in the definition space. Section 2 provides a thorough discussion and formulates simple conditions that help avoiding various undesirable "paradoxes" on the extended set of objects.

Fishburn (1975), motivated by the bounded utility discussion (Arrow, 1974; Ryan, 1974), was the first to provide a general extension of von Neumann-Morgenstern Expected Utility (EU) without restricting utility to be bounded. His conditions for the finitely additive case are complex and lack intuitive underpinning, as he recognized himself. His countably additive extension is simpler and can be subsumed to the general strategy proposed here (see section 5.1 for details). Wakker (1993) extended Rank Dependent Utility (RDU) and EU, both for risk and uncertainly[1] in a finitely additive context. Kothiyal et al. (2011) similarly extended the prospect theory functional. The results provided in this paper are structurally and logically more general than those mentioned above. Kopylov (2011) gave an alternative foundation for Savage's EU with countably additive subjective probability. He did not use Savage's P6 and P7, which makes his representation the most elegant axiomatization of EU proposed so far. His setup is considerably different from the

[1]RDU for uncertainty is often called Choquet Expected Utility

one pursued here and the two approaches are logically independent.

In what follows, section 2 defines the framework for the extension theorems. Those are formulated as general and abstract results in section 3. The rest of the paper (sections 4-6), use these results to extend in one blow the most famous theorems for risk and uncertainty. Section 4 is concerned with the state space setup and extends Savage's EU, Choquet EU and prospect theory (PT). Section 5 deals with risk, and provides the extensions of von von Neumann-Morgenstern EU, Rank Dependent Utility (RDU), Betweenness (Fishburn, 1983) and Disappointment Aversion (Gul, 1991). Section 6 deals with horse-lotteries and extends the EU of Anscombe and Aumann (1963), Choquet EU of Schmeidler (1989) and MaxMin EU of Gilboa and Schmeidler (1989).

It goes without saying that, even though the proposed method is illustrated for risk and uncertainty, the main theorems can be used equally well for extensions in other domains such as intertemporal choice, welfare, or multi-criteria decision making.

2 Definitions

The primitive sets of our setup are:

- $\mathscr{F}$ – a set of all objects of interest to the decision maker, denoted f, h, l, on which a *(preference) relation* $\succcurlyeq$ is assumed.[2] Derived relations $\preccurlyeq$, $\succ$,$\prec$ and $\sim$ on $\mathscr{F}$ are defined in the standard way. Usually h refers to"high", and l refers to "low".
- $\mathscr{F}^s \subseteq \mathscr{F}$ is a set of *simple* objects[3], denoted f^s, h^s, l^s, and sometimes just s.
- $\mathscr{F}^c \subseteq \mathscr{F}^s$ is a set of *constant* objects, being the "simplest" objects available[4], denoted $\alpha, \beta, \gamma, \mu, \nu$ and sometimes c (constant).

The goal of this paper is to give extensions of behavioral representation from simple ($\mathscr{F}^s$) to bounded objects ($\mathscr{F}^b$). A representation is a finitely valued function $F : \mathscr{F} \to \mathbb{R}$ such that $h \succcurlyeq\Leftrightarrow F(h) \geq F(l)$. A primitive concept of our analysis is a dominance relationship on $\mathscr{F}$:

[2] Examples of $\mathscr{F}$ are the set of acts as in Savage (1954) setup, probability measures as in Neumann and Morgenstern (1944), state-lotteries as in Anscombe and Aumann (1963), streams of outcomes over time as in Koopmans (1972) , or just sets as in Fishburn (1972).

[3] Can be simple acts (taking finitely many values), simple probability measures (distributions with finite support), or income streams with finitely many prices. In the context of sets, $\mathscr{F}^s$ may represent finite sets.

[4] Can be constant acts, sure probability distribution, or just singleton sets.

Definition 2.1. The *dominance relationship* $\trianglerighteq$ is a binary relation on $\mathscr{F}$ that is *complete* on $\mathscr{F}^c$,i.e., for all $\alpha, \beta \in \mathscr{F}^c$ $\alpha \trianglerighteq \beta$, or $\beta \trianglerighteq \alpha$, or both. $l \trianglelefteq h$ denotes $h \trianglerighteq l$.

By completeness, $\trianglerighteq$ is reflexive on $\mathscr{F}^c$, i.e., $a \trianglerighteq a$. It will serve as a placeholder for well known partial orders such as *pointwise dominance, (first order) stochastic dominance* or *conditional dominance.*

Definition 2.2. $f \in \mathscr{F}$ is *bounded above* if there exists $\alpha \in \mathscr{F}^c$ such that $\alpha \trianglerighteq f$, and similarly f is *bounded below* if there exists $\beta \in \mathscr{F}^c$ such that $\beta \trianglelefteq f$. f is *unbounded* above/below if such α, β do not exist. We say that an object is *bounded*, it is bounded from below and above.

Using the above definition, a set of major importance for the rest of the paper can be formally defined:

- $\mathscr{F}^b \subseteq \mathscr{F}$ is the subset of **bounded** objects.

We assume $\mathscr{F}^s \subseteq \mathscr{F}^b$, and the intuitive relation $\mathscr{F}^c \subseteq \mathscr{F}^s \subseteq \mathscr{F}^b \subseteq \mathscr{F}$ holds throughout the paper.

Definition 2.3. $\succcurlyeq$ satisfies *monotonicity* if $f \succcurlyeq l$ whenever $f \trianglerighteq l$.

The next property is essentially richness condition on $\mathscr{F}^s$. For every two objects, with at least one of them bounded from one side, there exists a special simple object in between:

Definition 2.4. *Simple approximation* holds on $\mathscr{S} \subseteq \mathscr{F}$ if for all $f \in \mathscr{S}$:

- For $h \in \mathscr{S}, h \succ f$, there exists a simple object $f_+^s \in \mathscr{F}^s$ such that

  $$h \succ f_+^s \trianglerighteq f$$

  whenever f is bounded above.

- For $l \in \mathscr{S}, l \prec f$, there exists a simple object $f_-^s \in \mathscr{F}^s$ such that

  $$l \prec f_-^s \trianglelefteq f$$

  whenever f is bounded below.

The above condition simply says, that in every preference interval[5], weakly preferred to f, there must exists a simple object dominating f (similarly for

[5] Preference interval (l, h) contains all f such that $l \prec f \prec h$. The interval $[l, h)$ is weakly preferred to l and it contains all $f \sim l$.

the reversed preferences). Simple approximation is central in practical measurements. In our essentially discrete world, every complex and "continuous" object is approximated by simpler, well understood, and precisely measured objects such as rulers, and units of mass and time. Without this approximation condition, any practical measurement of complex and "continuous" objects would be impossible.

The approximation is not performed by any arbitrary objects from $\mathscr{F}^s$, but rather, by simple objects that bear a special dominance relation $\trianglerighteq$ to f. Since in applications this dominance relationship has always a precise and intuitive meaning, the measurement procedure resulting from the above approximation has also a natural and normative meaning.[6] For example, to measure an area of an irregular object, one would not use arbitrary simple shapes, but only those that "fit" inside the irregular shape, and only those that approximate it from outside, resulting in an inner and outer measure of the object. Another example is the definition of a Lebesgue integral of a bounded function f, which is defined as the supremum and infimum over simple objects monotonically dominated and dominating f.

Simple approximation will be used to extend representations from $\mathscr{F}^s$ to $\mathscr{F}^b$. For the extension from $\mathscr{F}^b$ to $\mathscr{F}$, a procedure to approximate unbounded objects by bounded ones is required.

Definition 2.5. *Truncation from above (with respect to* $\trianglerighteq$*)* is an operator $^\wedge : \mathscr{F} \times \mathscr{F}^c \longrightarrow \mathscr{F}$ such that $\nu \trianglerighteq f^{\wedge\nu}$ and $f^{\wedge\nu} \trianglerighteq s \Rightarrow f \trianglerighteq s$ for $s \in \mathscr{F}^s$. Similarly, *truncation from below* is an operator $_\vee : \mathscr{F} \times \mathscr{F}^c \longrightarrow \mathscr{F}$ such that $\mu \trianglelefteq f_{\vee\mu}$ and $f_{\vee\mu} \trianglelefteq s \Rightarrow f \trianglelefteq s$ for $s \in \mathscr{F}^s$. Moreover, $(f^{\wedge\nu})_{\vee\mu} = (f_{\vee\mu})^{\wedge\nu}$, that is, the order of truncations doesn't matter. Here $f^{\wedge\nu}$ and $f_{\vee\mu}$ are short for $^\wedge(f,\nu)$ and $_\vee(f,\mu)$.

Thus, a truncation $f^{\wedge\nu}$ is bounded above and, in the spirit of transitivity, all the simple objects s dominated by a truncated object $f^{\wedge\nu}$ are also dominated by the original f. We will encounter two types of truncation operators – outcome-wise truncation in section 4 and probabilistic truncation in section 5. Outcome-wise truncation (Def. 4.7) reduces unbounded acts to bounded ones. Probabilistic truncation (Def. 5.4) maps probability measures with unbounded support into measures with bounded support. Yet another example of truncation, *conditional truncation*, is a condition used by Fishburn (1975) for his extension (see our Def. 5.10).

The following definition describes a natural relationship between truncation and preference $\succcurlyeq$:

[6] Pointwise and stochastic dominance are examples.

Definition 2.6. *Truncation monotonicity* holds if for all $f \in \mathscr{F}$ and all $\nu, \mu \in \mathscr{C}$

$$f_{\vee\mu} \succcurlyeq f \succcurlyeq f^{\wedge\nu}$$

whenever these truncations are defined.

Similarly to simple approximation, which enables us to approximate bounded objects with simple ones, the next condition makes it possible to approximate unbounded objects with bounded ones:

Definition 2.7. A preference $\succcurlyeq$ satisfies *truncation approximation*[7] on $\mathscr{S} \subseteq \mathscr{F}$ if for all $f \in \mathscr{S}$ the following conditions are satisfied

- For $h \in \mathscr{S}$, $h \succ f$, there exists a truncation $f_{\vee\mu}$ of f such that

  $$h \succ f_{\vee\mu} \succcurlyeq f$$

  whenever f is unbounded below.

- For $l \in \mathscr{S}$, $l \prec f$, there exists a truncation $f^{\wedge\nu}$ of l such that

  $$l \prec f^{\wedge\nu} \preccurlyeq f$$

  whenever f is unbounded above.

Truncation approximation suggests that the truncations are dense enough to ensure that there are no gaps in the measurement. There must not exist an object h such that it is valued higher than all truncations $f^{\wedge\nu}$, but lower than f itself. Truncation approximation is always imposed for the unbounded side of an object f, and for it to be satisfied, $\mathscr{F}$ needs to contain enough truncated objects. Hence, the following richness assumption:

Definition 2.8. $\mathscr{F}$ is *truncation rich* if for every $\sigma \in \mathscr{F}^c$, $f \in \mathscr{F} \backslash \mathscr{F}^b$ there exists $\nu \in \mathscr{F}^c, \nu \trianglerighteq \sigma$ such that $f^{\wedge\nu} \in \mathscr{F}$, and there exists $\mu \in \mathscr{F}^c, \mu \trianglelefteq \sigma$ such that $f_{\vee\mu} \in \mathscr{F}$.

One of the most essential behavioral conditions needed in our extensions is *non-confoundness*. This condition states that two objects cannot get confounded because there must always be a simple object in between:

[7]This condition was first proposed by Wakker's (1993) under the name *truncation continuity*. Wakker's version is slightly weaker, but more complex, requiring f to be simple. Given that he also requires simple equivalence (for $f \in \mathscr{F}, \exists f^s \in \mathscr{F}^s$ such that $f^s \sim f$) his condition immediately implies mine. A more intuitive name is used here mainly because of its similarity to simple approximation.

Definition 2.9. *Non-confoundness* holds on $\mathscr{S} \subset \mathscr{F}$ if for every $h, l \in \mathscr{S}, h \succ l$, there exists simple objects $s_h, s_l \in \mathscr{F}^s$

$$\begin{aligned} h &\succcurlyeq s_h \succ l \\ &\text{and} \\ h &\succ s_l \succcurlyeq l \end{aligned} \tag{2.1}$$

By its nature, non-confoundness is distinct from simple approximation and truncation approximation, which both impose stronger conditions in terms of $\trianglerighteq$. Non-confoundness is not based on $\trianglerighteq$, and objects s_h, s_l need not bear any special relationship to h or l.

Non-confoundness is immediate when *simple equivalence*[8] holds, which it does for virtually all representation theorems in the literature. Hence, it does not impose an extra restriction relative to existing results. Also, non-confoundness is trivially satisfied on $\mathscr{F}^s$ and immediately follows from simple approximation on $\mathscr{F}^b$. Truncation approximation implies non-confoundness for all pairs $h, l \in \mathscr{F}$ except the case when h is unbounded below and l is unbounded above. This weak form of "confoundness" is discussed further in the appendix A.

The main difficulty in applications is, usually, to prove simple approximation on $\mathscr{F}^b$. The following Lemma shows under which conditions non-confoundness implies simple approximation.

Lemma 2.10. *Assume that F represents $\trianglerighteq$ on $\mathscr{F}^s$. If $\trianglerighteq$-monotonicity and non-confoundness hold on $\mathscr{S} \in \mathscr{F}$, and for each $f \in \mathscr{S}$ and $\epsilon \in \mathbb{R}$, there exist simple functions $s^+, s^- \in \mathscr{F}^s$ such that $s^+ \trianglerighteq f \trianglerighteq s^-$ and $F(s^+) - F(s^-) < \epsilon$, then simple approximation holds on $\mathscr{S}$.*

In applications, the sandwiching condition of the above Lemma is often straightforward to prove. The essential role of non-confoundness, in terms of the measurement functional F, is that once $h \succ l$, $F(h) = F(l)$ cannot hold. Here is a simple example illustrating the potential problems if non-confoundness is not imposed.

Example 2.11. [Complexity Aversion]
Let $\mathscr{F} = \mathscr{F}^b$ contain all bounded real functions $f : (0,1) \to [0,1]$ Assume that $E(f) := \int_{(0,1)} f(\omega) d\omega$ represents $\succcurlyeq$ on the set of simple functions $\mathscr{F}^s$. Also E represents $\succcurlyeq$ on the set of non-simple functions $\mathscr{F}^b \backslash \mathscr{F}^s$. But, if $s \in \mathscr{F}^s$ and $b \in \mathscr{F}^b \backslash \mathscr{F}^s$ then $s \succ b$ if $E(s) \geq E(b)$, and $s \prec b$ if $E(s) < E(b)$. That is, the decision maker is averse to complex objects and always selects simple ones when expectations are the same, $E(s) = E(b)$.

[8] For $f \in \mathscr{F}, \exists f^s \in \mathscr{F}^s$ such that $f^s \sim f$

It is clear that the preference, constructed in this way, has a lexicographic nature and cannot be represented by any real functional.[9]

In the above example $\succcurlyeq$ is complete and transitive, and simple approximation and non-confoundness hold on $\mathscr{F}^s$ and on $\mathscr{F}^b \backslash \mathscr{F}^s$, but not on $\mathscr{F}^b$. This anomalous case is naturally avoided in the presence of non-confoundness.[10]

The reader might wonder if the symmetric version of non-confoundness 2.9 condition – *for any* $h \succ l$ *there exist* $s \in \mathscr{F}^s$, *such that* $h \succcurlyeq s \succcurlyeq l$ – is enough to preclude the above problem. It is straightforward to verify that this weaker condition holds in the above example, and thus cannot preclude the confoundness.

Non-confoundness can also play an explicit role in restricting the set of available objects to only those for which $F(\cdot)$ is finite. This can be usually achieved by imposing a richness conditions on $\mathscr{F}$ that are, unfortunately, representation dependent.[11] As the aim of this paper is generality and simplicity, a less strenuous strategy is adopted here in order to achieve finite-valued representations $F(\cdot)$. The following condition imposes boundedness restrictions on maximal and minimal elements of $\mathscr{F}$, if such exist.

Axiom 2.12. [Boundedness of the extremal elements] *If there exist a maximal element* $\overline{f} \in \mathscr{F}$, *such that* $\overline{f} \succcurlyeq f, \forall f \in \mathscr{F}$, *then* $\overline{f}$ *is bounded above. Similarly, if there exist a minimal element* $\underline{f} \in \mathscr{F}$, *such that* $\underline{f} \preccurlyeq f\ \forall f \in \mathscr{F}$, *then* $\underline{f}$ *is bounded below.*

It is easy to see that non-confoundness and the above axiom, together with monotonicity, imply that the objects with infinite value of F cannot exist in the domain $\mathscr{F}$. To see this, assume that F is finitely valued on $\mathscr{F}^b$ and represents $\succcurlyeq$ on $\mathscr{F}$. Assume that $F(f) = \infty$ for some unbounded f. Then by Axiom 2.12, f should be bounded above, or there must exists $f' \succ f$. The first case leads to $F(f) < \infty$, a contradiction. The second case, by non-confoundness implies that there exists simple $f^s \succcurlyeq f$, which again leads to contradiction $F(f) < \infty$.

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[9] If such an F, which represents $\succcurlyeq$, would exists, then it would be possible to construct an uncountably many open intervals $(F(b), F(s))$. This contradicts the topological separability of the real line.

[10] A version of example 2.11, which might be called *unboundedness aversion*, can be easily constructed for unbounded objects, with simple approximation holding on $\mathscr{F}^b$, truncation approximation on $\mathscr{F}$, and non-confoundness holding on $\mathscr{F}^b$ and $\mathscr{F} \backslash \mathscr{F}^b$, but not on the whole $\mathscr{F}$.

[11] Such a richness assumption can be informally stated as follows – *if* $f \in \mathscr{F}$ *then a (representation specific) modification* f' *of* f, *such that* $f' \not\sim f$, *must also be in* $\mathscr{F}$. Consequently, if $F(f) = \infty$, and $f' \in \mathscr{F}$ exists such that $F(f') = \infty$, but $f' \not\sim f$, by non-confoundness a contradiction is achieved, and such f should not be in $\mathscr{F}$ in the first place. Thus, $F(f) < \infty$ for all $f \in \mathscr{F}$ and St. Petersburg prospects cannot be constructed.

Finally, we turn to the properties of the measurement functional F. Our aim is to find necessary and sufficient conditions on $\succcurlyeq$, such that F represents $\succcurlyeq$ on $\mathscr{F}$, that is $h \succcurlyeq l$ if and only if $F(h) \geq F(l)$. F is *$\trianglerighteq$-monotonic* (or monotonic with respect to $\trianglerighteq$) if $F(h) \geq F(l)$ whenever $h \trianglerighteq l$. F is *truncation monotonic* (or monotonic with respect to truncation) if $F(f) \geq F(f^{\wedge\nu})$ and $F(f) \leq F(f_{\vee\mu})$. The next property requires a well-behavedness of the measurement functional F on a subset $\mathscr{S}$ of $\mathscr{F}$:

Definition 2.13. $F : \mathscr{F} \to \mathbb{R}$ satisfies *simple denseness* on the set $\mathscr{S} \subseteq \mathscr{F}$ if for every $h, l \in \mathscr{S}$ and every ϵ_1, ϵ_2, $F(l) < \epsilon_1 < \epsilon_2 < F(h)$ there exist $f^s \in \mathscr{F}^s$ such that $\epsilon_1 \leq F(f^s) \leq \epsilon_2$.

Simple denseness ensures that elements of $\mathscr{F}^s$ penetrate a subset of $\mathbb{R}$ similarly to how rational numbers $\mathbb{Q}$ penetrate the real line $\mathbb{R}$. This condition holds when $F(\mathscr{F}^s)$ is a dense subset of an interval in $\mathbb{R}$. It follows from the other well known richness conditions as those for state spaces (Savage, 1954, mainly by $P6$), outcome spaces (Debreu, 1959; Wakker, 1989, from topological connectedness) and mixture-spaces (Neumann and Morgenstern, 1944, convexity and vNM-continuity). Thus, it adds no restrictions to existing theorems.

3 Main Theorems

Our first theorem is the simplest extension from simple to bounded objects:

Theorem 3.1. [Extension from $\mathscr{F}^s$ to $\mathscr{F}^b$] *Assume that F represents $\succcurlyeq$ on $\mathscr{F}^s$ and that simple denseness holds on $\mathscr{F}^s$.*

There exists an unique extension of F that represents $\succcurlyeq$ on $\mathscr{F}^b$, and is monotonic with respect to $\trianglerighteq$, defined as

$$F^*(f) = \sup\{F(f^s) : f^s \trianglelefteq f, f^s \in \mathscr{F}^s\} = \inf\{F(f^s) : f^s \trianglerighteq f, f^s \in \mathscr{F}^s\} \tag{3.1}$$

if and only if $\succcurlyeq$ satisfies the following conditions on $\mathscr{F}^b$:

(i) weak ordering

(ii) $\trianglerighteq$-monotonicity

(iii) simple approximation

When Theorem 3.1 is used in applications, and F can be explicitly defined on $\mathscr{F}^b$, it is also necessary to verify that $F = F^*$ on $\mathscr{F}^b$. This is straightforward when F is monotonic with respect to $\trianglerighteq$ on $\mathscr{F}^b$. Then, it follows from (3.1) that $F = F^*$ on $\mathscr{F}^b$. When F is not explicitly defined on $\mathscr{F}^b \backslash \mathscr{F}^s$ (the betweenness model of Dekel (1986) is such an example), then Eq. (3.1) can be taken as a formal definition.

The only structural assumption used in the above theorem is simple denseness on $\mathscr{F}^s$. This condition ensures that $(F, \mathscr{F}^s)$ is a proper measurement pair and F cannot assign values to objects in $\mathscr{F} \backslash \mathscr{F}^s$ "far away" from those in $\mathscr{F}^s$.

An important case arises when $\mathscr{F}$ contains only objects that are unbounded from at most one side. Additive and non-additive measures are examples. In particular, when F is an additive measure, the following extension is a version of the Caratheodory extension theorem. Denote by $\mathscr{F}^o$ a set of objects that contains unbounded objects at most from one side.

Theorem 3.2. **[Extension from $\mathscr{F}^b$ to $\mathscr{F}^o$]** *Assume that F^* represents $\succcurlyeq$ on $\mathscr{F}^b$, simple denseness holds on $\mathscr{F}^s$, and that extremal boundedness 2.12 holds.*

There exists an unique extension of F^* that represents $\succcurlyeq$ on $\mathscr{F}^o$, and is truncation-monotonic and $\trianglerighteq$-monotonic, defined by

$$F^{**}(f) = \begin{cases} F^*(f) \text{ for } f \in \mathscr{F}^b \\ \sup_{l^s \trianglelefteq f} F^*(l^s) \text{ for } f \text{ bounded below} \\ \inf_{h^s \trianglerighteq f} F^*(h^s) \text{ for } f \text{ bounded above} \end{cases} \tag{3.2}$$

if and only if $\succcurlyeq$ satisfies the following conditions on $\mathscr{F}^o$:

(i) weak ordering

(ii) $\trianglerighteq$-monotonicity

(iii) simple approximation

(iv) non-confoundness

For the extension in full generality, when $\mathscr{F}$ contains objects unbounded from both sides, the truncation approximation becomes a necessary condition:

Theorem 3.3. **[Extension from $\mathscr{F}^b$ to $\mathscr{F}$]** *Assume that F^* represents $\succcurlyeq$ on $\mathscr{F}^b$, simple denseness holds on $\mathscr{F}^b$, truncation richness holds on $\mathscr{F} \backslash \mathscr{F}^b$, and that extremal boundedness 2.12 holds.*

There exists an unique extension of F^* that represents $\succcurlyeq$ on $\mathscr{F}$, and is truncation-monotonic and $\trianglerighteq$-monotonic, defined as

$$F^{**}(f) = \begin{cases} F^*(f) \text{ for } f \in \mathscr{F}^b \\ \sup_\nu F^*(f^{\wedge\nu}) \text{ for } f \text{ bounded below} \\ \inf_\mu F^*(f_{\vee\mu}) \text{ for } f \text{ bounded above} \\ \sup_\nu \inf_\mu F^*(f^{\wedge\nu}_{\vee\mu}) = \inf_\mu \sup_\nu F^*(f^{\wedge\nu}_{\vee\mu}) \\ \qquad \text{for } f \text{ unbounded from both sides} \end{cases} \tag{3.3}$$

if and only if $\succcurlyeq$ satisfies the following conditions on $\mathscr{F}$

(i) weak ordering,

(ii) $\trianglerighteq$-monotonicity,

(iii) truncation monotonicity

(iv) truncation approximation

(v) non-confoundness

The comment following Theorem 3.1 also applies here. If F can be formally defined on $\mathscr{F}\backslash\mathscr{F}^b$, then it must be verified that F and F^{**} coincide. The easiest way to prove this on the set of non-extremal objects, is first to ensure that F implies all the conditions of the theorem. Then by the uniqueness result we must have $F = F^{**}$ on the non-extremal set $\tilde{\mathscr{F}}$. For other points (maximal or minimal unbounded objects), one has to prove that F is indeed a supremum (infimum) of values of truncations. In all applications this follows straightforwardly from the definition of F.

Sometimes it is not possible to characterize F in terms of known functions or integrals. Level dependent integrals and level dependent Choquet integrals, discussed and axiomatized by Wakker and Zank (1999), and Chew and Wakker (1996) are such examples. Then Eq. (3.3) is used directly as the definition of F^{**}.

Note that simple approximation is not used in Theorem 3.3. Given the representation on $\mathscr{F}^b$, truncation approximation is enough to approximate all unbounded objects. The following lemma shows that simple approximation and the conditions of the above theorem, except truncation monotonicity, readily imply truncation monotonicity.

Lemma 3.4. *Assume that $\succcurlyeq$ is a weak order on $\mathscr{F}$ and $\succcurlyeq$ satisfies $\trianglerighteq$-monotonicity, simple approximation on $\mathscr{F}^b$, truncation approximation and weak non-confoundness. Then truncation monotonicity holds.*

An immediate implication of Theorems 3.1 and 3.3 and Lemma 3.4 is our main theorem:

Theorem 3.5. [Extension from $\mathscr{F}^s$ to $\mathscr{F}$] *Assume that F represents $\succcurlyeq$ on $\mathscr{F}^s$, simple denseness holds on $\mathscr{F}^s$, truncation richness holds on $\mathscr{F} \setminus \mathscr{F}^b$, and that extremal boundedness 2.12 holds.*

There exists an unique extension of F that represents $\succcurlyeq$ on $\mathscr{F}$, and is $\trianglerighteq$-monotonic and truncation-monotonic, and is defined as in Eq. (3.3), if and only if $\succcurlyeq$ satisfies the following conditions on $\mathscr{F}$:

(i) weak ordering

(ii) $\trianglerighteq$-monotonicity

(iii) simple approximation on $\mathscr{F}^b$

(iv) truncation approximation

(v) weak non-confoundness

All four main theorems provided in this section can serve as add-on tools to extend existing models in the literature. The imposed conditions, except simple approximation, usually follow immediately from the assumptions of the original representations on $\mathscr{F}^s$. The rest of the paper consists of applications of the above theorems.

4 State Space

This section gives extensions of several well known models that use the common state space framework of Savage (1954).

Assumption 4.1. [State space Setup] *Ω is the set of states and $\mathscr{C}$ is the set of consequences. $\mathscr{A}_\Omega$ is an algebra of subsets of Ω called events, and $\mathscr{A}_\mathscr{C}$ – a class of subsets of $\mathscr{C}$ containing all singleton consequences.*[12] *$\mathscr{F}$ is a set of measurable functions from Ω to $\mathscr{C}$ called acts. $\mathscr{F}^s \subseteq \mathscr{F}$ is the set of simple functions*[13]*, and need not contain all simple functions. $\mathscr{F}^c$ is a set of constant functions. $\mathscr{F}$ is truncation rich and extremal boundedness 2.12 holds.*

[12] The assumption that $\mathscr{A}_\mathscr{C}$ contains all elements of $\mathscr{C}$ is made for convenience to avoid some measure theoretic complications concerning the distinction between step-functions and simple-functions, and can be easily dispensed with. With the above assumption step and simple functions are the same.

[13] Recall that simple functions are measurable functions taking finitely many values.

As the state space framework 4.1 does not impose any richness conditions on Ω or $\mathscr{C}$, it can be shared by three prominent setups, that of Savage (1954), imposing richness on Ω, that of Debreu (1959) and Wakker (1989), imposing richness of $\mathscr{C}$, and that of Anscombe and Aumann (1963) also imposing convexity on $\mathscr{C}$. Except truncation richness there are no other richness requirement on $\mathscr{F}$. Particularly, $\mathscr{F}^s$ need not contain *all* simple functions as it is commonly assumed in the literature.

In what follows, constant objects in $\mathscr{F}^c$ are, for simplicity, identified with elements of the consequence space $\mathscr{C}$. The standard notation $l_A h$ is used to denote an act which equals l on $A \in \mathscr{A}$ and h on $\Omega \backslash A$, for any $l, h \in \mathscr{F}$. The central behavioral principle of Savage's expected utility theory is:

Definition 4.2. $\succcurlyeq$ satisfies the *sure-thing principle* on $\mathscr{S} \subseteq \mathscr{F}$ if

$$h_A f \succcurlyeq l_A f \Leftrightarrow h_A f' \succcurlyeq l_A f'$$

wherever all the above acts are in $\mathscr{S}$.

When the sure-thing principle holds, we can unambiguously write $h_A \succcurlyeq l_A$, if there exists f such that $h_A f \succcurlyeq l_A f$, with an obvious meaning that h is preferred to l conditionally on event A being true. For the statement of Savage's *EU* theorem, our partial relation $\trianglerighteq$ takes the form of:

Definition 4.3. *Conditional dominance* $\trianglerighteq_c$ is defined as follows:

- $h \trianglerighteq_c f$ if there exists a finite partition of Ω, $\{A_1, \dots A_n\}$ such that $(\forall A_i, \forall \omega \in A_i : h(\omega)_{A_i} \succcurlyeq f_{A_i})$
- $f \trianglerighteq_c l$ if there exists a finite partition of Ω, $\{A_1, \dots A_n\}$ such that $(\forall A_i, \forall \omega \in A_i : f_{A_i} \succcurlyeq l(\omega)_{A_i})$.

This leads to Savage's monotonicity condition $P7$[14]:

Definition 4.4. $\succcurlyeq$ satisfies *conditional monotonicity* on $\mathscr{F}$ if for all $h, l \in \mathscr{F}$

$$h \trianglerighteq_c l \Rightarrow h \succcurlyeq l \quad .$$

In the presence of additional assumptions, such as countable additivity of the subjective probability or connectedness of the outcome space, a simpler, pointwise monotonicity condition is enough for the *EU* representation:

[14] Wakker (1993) used weaker, conditional monotonicity and sure thing principle conditions. The exposition and all the proofs of the current section stay virtually unchanged if Savage's conditions are replaced by Wakker's ones. Wakker's results are presented in the appendix.

Definition 4.5. *Pointwise dominance* $\trianglerighteq_p$ on $\mathscr{S} \subseteq \mathscr{F}$, is defined as follows:

$$h \trianglerighteq_p l \Leftrightarrow h(\omega) \succcurlyeq l(\omega), \forall \omega \in \Omega$$

for all $l, h \in \mathscr{S}$.

And monotonicity becomes:

Definition 4.6. $\succcurlyeq$ satisfies *pointwise monotonicity* on $\mathscr{S} \subseteq \mathscr{F}$ if for all $l, h \in \mathscr{S}$

$$h \trianglerighteq_p l \Rightarrow h \succcurlyeq l \quad .$$

The abstract truncation operator from Def. 2.5 takes the form of the *outcome-wise truncation*:

Definition 4.7. $f^{\wedge\nu}$ is an *outcome-wise truncation from above* of f if $f^{\wedge\nu}(\omega) = f(\omega)$ for $\{\omega : f(\omega) \trianglelefteq \nu\}$ and $f^{\wedge\nu} = \nu$ otherwise. Similarly an *outcome-wise truncation from below* $f_{\vee\mu}$ equals $f(\omega)$ on $\{\omega : f(\omega) \trianglerighteq \mu\}$ and $f_\mu = \mu$ otherwise.

For both $\trianglerighteq_c$ and $\trianglerighteq_p$, outcome-wise truncation satisfies the requirements of the generic Def. 2.5 of truncation.

Definition 4.8. Event $A \in \mathscr{A}$ is *null* if for all $h, l, f \in \mathscr{F}$, $h_A f \sim l_A f$ holds.

Given an expected utility representation, null events are precisely those with subjective probability zero.

4.1 Expected Utility (Savage, 1954)

Expected Utility with respect to subjective probability measure P on $\mathscr{A}_\Omega$ and real valued utility $U : \mathscr{C} \to \mathbb{R}$ is defined as the Lebesgue integral of $U(f)$:

$$EU(f) = \int_\Omega U(f) dP \quad .$$

In order to prove the existence of a finitely additive subjective probability on $\mathscr{A}_\Omega$, Savage (1954, §3.4 p.42-43) required $\mathscr{A}_\Omega$ to be a σ-algebra. His axioms imply boundedness of utility (see Fishburn, 1970, ch.14.5; p.206). As noted before, to avoid this limitation, one can explicitly restrict the set $\mathscr{F}$ (to the set of all simple or bounded acts for example), or to impose additional behavioral conditions to implicitly restrict $\mathscr{F}$.

Wakker (1993) extended Savage's theorem in a finitely additive framework and used simple equivalence and truncation-continuity to limit the set $\mathscr{F}$ to objects with well defined and finite expected utility. Kopylov (2011) gave an extension of the Savage representation theorem for unbounded acts by means of a *strong monotone continuity* condition, which implies among other things, countable additivity of the subjective probability P. An important advantage of countable additivity is that conditional monotonicity ($P7$) is no longer required, and the simpler pointwise monotonicity suffices. Kopylov's work is based on Kopylov (2007) which provided an axiomatization of subjective expected utility in a framework that is structurally considerably more general than Savage's. Kopylov used new structures, *mosaics*, which need not be closed under unions, intersections or even set difference. This opened interesting opportunities in modeling sources of uncertainty (Abdellaoui et al., 2011). In his original work Kopylov (2007) differentiates between "risky" and ambiguous events and modeled the set of "risky" events as a mosaic.

This paper assumes, as did Kopylov (2011), that $\mathscr{A}_\Omega$ is an algebra and need not be a σ-algebra. Recall that Savage's representation theorem implies the existence of $A \in \mathscr{A}_\Omega$ such that $P(A) = q$ for every $q \in [0,1]$ which implies, under EU, that simple equivalence holds (i.e., for every $f \in \mathscr{F}$ there exists $f^s \in \mathscr{F}^s$ such that $f^s \sim f$). As a consequences of the weaker structural assumptions, Kopylov's representation on $\mathscr{F}^s$ does not imply simple equivalence. Instead, it follows from his theorem that for all $q, \epsilon \in (0,1]$ there exist $A^q_\epsilon \in \mathscr{A}_\Omega$ such that $P(A^q_\epsilon) \in [q-\epsilon, q+\epsilon]$ (or the image of P is dense in $(0,1)$. Kopylov called a probability measure that satisfies this property *finely ranged*.

In addition to the aforementioned structural relaxation of the Savage's setup, the extensions proposed here are also more general in a logical sense. The following theorems can handle any set $\mathscr{F}$ of acts with all $EU(f), f \in \mathscr{F}$, finite irrespective of whether U is bounded or not.

In what follows, two alternative extensions of Savage's (1954) theorem are stated, one for finitely and one for countably additive subjective probabilities. The relationship to Wakker and Kopylov's representations is discussed after.

Theorem 4.9. **[Extension of Finitely Additive EU]** *Under the state space setup 4.1, with $\mathscr{A}_\Omega$ an algebra, assume that EU represents $\succcurlyeq$ on $\mathscr{F}^s$ with respect to a finely ranged, finitely additive probability measure P and utility U.*

Then an unique EU representation holds on $\mathscr{F}$ with the same utility U and subjective probability P, if and only if the following conditions hold:

- *weak ordering*
- *conditional monotonicity*

- *truncation approximation*
- *non-confoundness*

Moreover EU is unique on $\mathscr{F}$, and if Axiom 2.12 holds, or $U(\mathscr{C})$ is bounded in $\mathbb{R}$, then $|EU(f)| < \infty$ for all $f \in \mathscr{F}$.

Proof of Theorem 4.9. First, I prove necessity. Weak ordering is obvious. Conditional monotonicity follows immediately because the value of an integral equals the sum of integrals on a finite partition of Ω. Truncation approximations follows easily from the definition of the Lebesgue integral.[15] For non-confoundness, let $h, l \in \mathscr{F}^b$ such that $EU(h) > EU(l)$. Take $c^h, c^l \in \mathscr{C}$ such that $U(c^h) \geq EU(h) > EU(l) \geq U(c^l)$. By fine-rangeness there exists an event E such that $EU(h) > EU(c^h_E c^l) > EU(l)$, and non-confoundness is proved. For $h, l \in \mathscr{F} \backslash \mathscr{F}^b$, by truncation richness and truncation monotonicity (which hold for Lebesgue integrals) there exist close enough truncated (from both sides, if needed) acts $b^h, b^l \in \mathscr{F}^b$ such that $EU(b^h) > EU(b^l)$ and $EU(b^h) > EU(l)$ and $EU(h) > EU(b^l)$, and we are in previous case of bounded h, l.

For sufficiency I will use Theorem 3.5. Hence it must be proved that simple denseness and simple approximation hold when $\trianglerighteq$ is the conditional dominance relationship as in Def. 4.3.

To prove simple denseness on $\mathscr{F}^s$, let $EU(l^s) < EU(h^s)$ for $f^s, h^s \in \mathscr{F}^s$. Take the outcomes $m, M \in \mathscr{C}$ to be common lower an upper bounds of l^s and h^s. By fine-rangeness of P, for all $\epsilon \in (EU(l^b), EU(h^b))$, here must exist, A^-_ϵ and A^+_ϵ such that $EU(l^s) < EU(m_{A^-_\epsilon} M) < \epsilon < EU(m_{A^+_\epsilon} M) < EU(h^s)$.

To prove simple approximation on $\mathscr{F}^b$, let $m \preccurlyeq M \in \mathscr{C}$ be bounds of $f \in \mathscr{F}^b$. Without loss of generality let $U(m) = 0$ and $U(M) = 1$. For each $n \in \mathbb{N}$, partition Ω into:

$$A_0 := \{w \in \Omega : 0 \leq u(f(w)) \leq \frac{1}{n}\} \tag{4.1}$$

and for all $1 < i \leq n - 1$,

$$A_i := w \in \Omega : \frac{i}{n} < u(f(w)) \leq \frac{i+1}{n} \quad . \tag{4.2}$$

Now set a small positive ϵ and take a set $E_i \subset A_i$ with probability $P(E_i) \in [\frac{i}{n} P(A_i) - \frac{\epsilon}{n^2}, \frac{i}{n} P(A_i)]$. This is always possible by fine rangednness of P.

[15] Lebesgue integral, is a sum of the integrals of the positive and negative parts of the function. For each part, the integral is a supremum of integrals of simple functions. As for any simple function, there exists a truncation pointwise dominating the simple function, the claim follows

As $EU(M_{E_i}m) = P(E_i)$ and EU holds on $\mathscr{F}^s$, $f_{A_i}(\omega) \succcurlyeq (M_{E_i}m)_{A_i}$ for all $\omega \in A_i$. Define s_n^- to be M on E_i and m on $A_i \backslash E_i$ for all i. By conditional-monotonicity, $f \succcurlyeq s_n^-$. Similarly, we can find events $T_i \subseteq A_i$ such that $P(T_i) \in [\frac{i+1}{n}P(A_i), \frac{i+1}{n}P(A_i) + \frac{\epsilon}{n^2}]$ and construct a function s_n^+ to be M on $\cup_i T_i$ and m otherwise. By conditional monotonicity, $s_n^+ \succcurlyeq f$. $EU(s_n^+) - EU(s_n^-) \leq \frac{1+\epsilon}{n}$. By lemma 2.10, simple approximation hold on $\mathscr{F}^b$.

It remains to prove that the F^{**} functional in Eq. (3.3) in Theorem 3.5 is indeed the EU integral

$$F^{**}(f) = \int_\Omega U(f(\omega))dP(w) \quad .$$

From the necessity part of the proof, EU satisfies all the conditions of Theorem 3.5, and by uniqueness of F^{**}, $F^{**} = EU^{**}$ on the whole set $\mathscr{F}$. Thus an unique and finitely valued EU holds on the whole $\mathscr{F}$.

□

Because of the importance of Savage's (1954) theorem, I state a generalized and self-contained statement of his result as a corollary of the main Theorem 3.5 of this paper. To illustrate the role of non-confoundness for the St. Petersburg paradox, instead of the Axiom 2.12, I use a weak richness condition, which is a version of a general condition portrait at the end of section 2:

Theorem 4.10. [Savage's EU with finitely additive probability] *Assume that state space setup 4.1 holds and $\mathscr{A}_\Omega$ is an algebra. An unique EU represents $\succcurlyeq$ on $\mathscr{F}$ with respect to a finely ranged probability measure P and monotone utility U, if and only if the following conditions hold:*

P1 *weak ordering*

P2 *sure-thing principleon $\mathscr{F}^s$*

P3 *if A is non-null then $\forall \alpha, \beta \in \mathscr{C} : \alpha \succcurlyeq \beta \Leftrightarrow \alpha_A \succcurlyeq \beta_A$*

P4 *if $\alpha \succ \beta$ and $\gamma \succ \delta$, $\alpha, \beta, \gamma, \delta \in \mathscr{C}$ then for events A,B: $[\alpha_A\beta \succcurlyeq \alpha_B\beta] \Leftrightarrow [\gamma_A\delta \succcurlyeq \gamma_B\delta]$*

P5 *$x \succ y$ for some $x, y \in \mathscr{F}$*

P6 *if for $f, h \in \mathscr{F}^s$, $f \succ h$ and $\alpha \in \mathscr{C}$, then there exists a partition $(A_1, \cdots, A_m)$ of Ω, $A_i \in \mathscr{A}_\Omega$, such that $\alpha_{A_i}f \succ h$ and $f \succ \alpha_{A_i}h$ for all i*

P7 *conditional monotonicity*

P'8 *truncation approximation on* $\mathscr{F}\backslash\mathscr{F}^b$

P'9 *non-confoundness*[16]

Proof of Theorem 4.10. Kopylov (2007) proved that conditions P1-P6 are necessary and sufficient for the SEU representation to hold on the set of simple acts $\mathscr{F}^s$ with finely-ranged subjective probability P and non-constant U. By Theorem 4.9, EU holds on the whole $\mathscr{F}$. To see that EU is finite-valued, assume for contradiction that there exists f such that $EU(f) = \infty$, and there must exist, $\alpha_A f, \beta_A f$ such that $\alpha \succ \beta$. As f_A is bounded, there must be that $EU(\beta_A f) = \infty$. By sure thing principle $\alpha_A f \succ \beta_A f$, and by non-confoundness there exist $s \in \mathscr{F}^s$, $s \succcurlyeq \beta_A f$, which implies $\infty > EU(s) \geq EU(\beta_A f)$ a contradiction. Thus EU is finite for all $f \in \mathscr{F}$. □

The above theorem and its counterpart B.3 (based on Wakker's (1993) weaker conditions), are the most general forms of Savage's theorem with finitely additive subjective probability so far. They are direct generalizations of the representations by Wakker (1993) who assumed, as did Savage, that A_Ω is a σ-algebra, and used simple-equivalence which is a more restrictive assumption than non-confoundness adopted here. His truncation-continuity condition is a version of the truncation approximation with f in Def. 2.7 confined to $\mathscr{F}^s$ instead of $\mathscr{F}$. Obviously, in the presence of his simple equivalence, his variant immediately implies mine. His other conditions are the same as in the above corollary.

Conditional monotonicity can be replaced by pointwise monotonicity if P is countably additive. Countable additivity can be captured by a *monotone continuity* behavioral condition (Arrow, 1971, p.???): for all acts $f^s, h^s \in \mathscr{F}^s$, outcomes $x \in \mathscr{C}$ and events $A_1, A_x, \ldots$ such that $A_i \to \emptyset$[17] and $f^s \succcurlyeq x_{A_i} h^s$ or $x_{A_i} f^s \succcurlyeq h^s$ for all i, then $f^s \succcurlyeq h^s$.

Theorem 4.11. [Extension of Countably Additive EU] *Under the state space setup 4.1 with* $\mathscr{A}_\Omega$ *an algebra, assume that EU represents* $\succcurlyeq$ *on* $\mathscr{F}^s$ *with respect to a finely ranged countably additive probability* P *and monotonic utility* U*.*

Then and unique EU represents $\succcurlyeq$ *on* $\mathscr{F}$ *with the same* U *and* P *if and only if the following conditions hold:*

(i) weak ordering,

[16]As a side note, non-confoundness would not be necessary here if we would have explicitly imposed simple approximation. Then non-confoundness would easily follow, and our main Theorem 3.5 could be directly used.

[17]$A_i \to \emptyset$ stands for $\lim_{n\to\infty} \cap_i^n A_i = \emptyset$.

(ii) pointwise monotonicity,

(iii) truncation approximation on $\mathscr{F}\backslash\mathscr{F}^b$,

(iv) non-confoundness.

Proof of Theorem 4.11. The proof is exactly the same as the proof of theorem 4.9 except for the simple approximation part which now uses pointwise monotonicity. Simple approximation on $\mathscr{F}^b$ is proved in what follows.

Let $m \preccurlyeq M \in \mathscr{C}$ be the bounds of $f \in \mathscr{F}^b$. Without loss of generality let $U(m) = 0$ and $U(M) = 1$. Let A_0 and A_i be as defined in equations (4.1) and (4.2).

Let $u_i = \inf(U(f(\omega)) : \omega \in A_i$. If there exists $\omega_i^* \in \Omega$ such that $U(f(\omega_i)) = u_i$ then define $s^-(\omega) = f(\omega_i^*)$, $\omega \in A_i$, and set $B_i = \emptyset$. If not, take a decreasing sequence of intervals $E_j = (u_i, \epsilon_j]$, obviously $E_j \to \emptyset$ and $f^{-1}(E_j) \to \emptyset$.

Because of countable additivity, there must exist ϵ_k such that $P(B_i) < \frac{1}{n^{i+1}(i+1)}$ where $B_i = \{\omega : U(f(\omega)) \in (u_i, \epsilon_k]\}$. Set $s^-(\omega) = m$ for $\omega \in B_i$.

The simple function $s^-(\omega_i) = s_i, \omega \in A_i$ is arbitrarily close in EU value to f and $f \trianglerighteq_p s^-$:

$$EU(f) - EU(s^-) = \tag{4.3}$$

$$\sum_{i=0}^{n-1} \int_{A_i \backslash B_i} \left[U(f(\omega)) - U(s_i^-)\right] dP + \sum_{i=0}^{n-1} \int_{B_i} \left[U(f(\omega)) - U(s_i^-)\right] < \tag{4.4}$$

$$\frac{1}{n} + \sum_{i=1}^{n} \frac{i}{n^i i} < \frac{1}{n} + \frac{1}{n-1} < \frac{2}{n-1} \tag{4.5}$$

We can similarly find a simple s^+ such that $s^+ \trianglerighteq_p f$ and $EU(s^+) - EU(f) < \frac{2}{n-1}$. Thus $EU(s^+) - EU(s^-) < \frac{4}{n-1}$. By lemma 2.10 simple approximation holds on $\mathscr{F}^b$.

□

For the above theorem, a counterpart to Theorem 4.10 obviously holds. It can be proved (Kopylov, 2011, Lemma 6) that in the presence of pointwise monotonicity and countable additivity, Savage's $P3$ holds and thus is no more necessary.

Kopylov (2011) imposed a stronger version of monotone-continuity[18] and an additional structural requirement of $\mathscr{A}_\Omega$ being *countably separable.* With

[18]Replace $A_i \to \emptyset$ by $A_i \to \omega \in \Omega$ in the above definition of monotone continuity.

these assumptions Kopylov managed to drop Savage's P6. His approach is incompatible with the direction taken in this paper. An integration of Kopylov's representation theorem into the current framework is left for future research.

4.2 EU and Choquet EU (Wakker, 1989)

As mentioned before, the state space setup 4.1 does not impose any richness conditions on the state space Ω, nor on the outcome space $\mathscr{C}$. Recall that Savage's theorem imposes richness of state space, mainly by $P6$. This section allows a general state space Ω, but adds a richness of outcomes assumption:

Axiom 4.12. **[Richness of outcomes]** *$\mathscr{C}$ is a connected topological space.*

If U is continuous, a necessary an sufficient condition for the above axiom, in all the results below, is that $U(\mathscr{C})$ is an interval. Hence, the above condition can be replaced by a more intuitive requirement that $U(\mathscr{C})$ is an interval in $\mathbb{R}$.

Richness of outcomes simplifies the behavioral conditions, most notably, by allowing pointwise monotonicity instead of a more complex conditional monotonicity. The above topological assumption is common in the literature and can be regarded as a generalization of de Finetti (1931, 1974) framework of subjective probability. De Finetti's approach allows an arbitrary state space, but requires a continuum of the outcome space.

A *capacity* is a set function $v : \mathscr{A}_\Omega \to [0, 1]$ that is monotonic with respect to set inclusion (i.e., $A \subseteq B \Rightarrow v(A) \leq v(B)$) and $v(\emptyset) = 0, v(\Omega) = 1$. The *Choquet integral* is defined as a sum of two Lebesgue integrals:

$$\int_c f dv = \int_{\mathbb{R}_+} v(s \in \Omega : f(s) \geq \tau) d\tau + \int_{\mathbb{R}_-} [v(s \in \Omega : f(s) \geq \tau) - 1] d\tau \quad (4.6)$$

Choquet expected utility (CEU) with respect to utility U and capacity v is defined as $CEU(f) = \int_c U(f(\omega)) dv$. Kobberling and PWakker (2003, Section 6) review the literature of CEU's axiomatizations in detail.

With $\trianglerighteq$ as pointwise dominance $\trianglerighteq_p$, and $\trianglerighteq$-monotonicity as pointwise monotonicity, the CEU representation follows:

Theorem 4.13. **[Extension of CEU]** *Under state space setup 4.1 and richness of outcomes assumption 4.12, assume that CEU represents $\succcurlyeq$ on $\mathscr{F}^s$ with respect to a capacity v and a monotone and continuous utility function $U : \mathscr{C} \to \mathbb{R}$. Then and unique CEU represents $\succcurlyeq$ on $\mathscr{F}$ with respect to the same v and U, if and only if the following conditions hold:*

(i) weak ordering

(ii) pointwise monotonicity

(iii) truncation approximation on $\mathscr{F}\backslash\mathscr{F}^b$

(iv) non-confoundness

Proof. Necessity follows as in the prof of Theorem 4.9. For sufficiency note, that by continuity of U and connectedness of $\mathscr{C}$, $U(\mathscr{C})$ must be an interval, thus, simple denseness is satisfied. All the assumptions of the Theorem 3.5, except simple approximation, hold.

In what follows, I prove simple approximation. For each integer m define $A_i, i \in \overline{0,m}$ as in eqs. (4.1) and (4.2). Now define

$$l^s_m(x) := \inf(f(x) : x \in A_i) \quad .$$

Similarly, define pointwise decreasing series of functions

$$h^s_m(x) := \sup(f(x) : x \in A_i) \quad .$$

Obviously $EU(h^s_m) - EU(l^s_m) \leq \frac{1}{m}$, and by Lemma 2.10 simple approximation is satisfied. By theorem 3.5 there exist an unique representation F^{**} as defined in Eq. (3.3). As CEU satisfies all the assumptions of the theorem, by uniqueness of F^{**}, $CEU = F^{**}$ on the whole set $\mathscr{F}$. □

Observation 4.14. Because EU is a subclass of CEU, the above theorem is also an EU extension.

Wakker (1989, sec. VI.5.1) used weak ordering, simple-continuity and comonotonic tradeoff consistency to provide a continuous CEU representation on $\mathscr{F}^s$. If comonotonic tradeoff-consistency is strengthened to tradeoff-consistency, an EU representation on $\mathscr{F}^s$ is obtained. These and other CEU/EU representations can be readily plugged into Theorem 4.13 to achieve full representations of CEU/EU on $\mathscr{F}$. Wakker (1993, Theorem 2.13) used pointwise monotonicity, simple equivalence and truncation continuity for his extension on $\mathscr{F}$. These conditions directly imply the conditions of the Theorem 4.13. Hence, Wakker's theorem is an immediate corollary of the above result. This makes Theorem 4.13 the most general representation of SEU with continuous utility so far. As in the case of extension of Savage's EU, U in the above theorem need not be bounded and any prospect with finite expected utility is allowed to be in the set $\mathscr{F}$.

4.3 Prospect Theory

Prospect Theory (Tversky and Kahneman (1992)) differs from CEU in that there are two distinct capacities, v^+ and v^-, one for gains and one for losses, and utility for losses incorporates "loss aversion":

$$PT(f) = \int_{\mathbb{R}_+} v^+(s \in \Omega : U(f(s)) \geq \tau) d\tau - \int_{\mathbb{R}_-} v^-(s \in \Omega : U(f(s)) \leq \tau) d\tau \tag{4.7}$$

Gains and losses are defined with respect to a special outcome $r \in \mathscr{C}$, a *reference point*. Kothiyal et al. (2011) provided the extension theorems of PT for uncertainty and risk. The next theorem is a generalization of their results as it does not require the existence of certainty equivalents[19]:

Theorem 4.15. [Extension of PT] *Under the state space setup 4.1 and the richness of outcomes assumption 4.12, assume that PT represents $\succcurlyeq$ on $\mathscr{F}^s$ with respect to capacities v^+, v^- and a monotone and continuous utility function $U : \mathscr{C} \to \mathbb{R}$, with $U(r) = 0$. Then PT represents $\succcurlyeq$ on $\mathscr{F}$ with respect to the same v^+, v^- and U, if and only if the following conditions hold on $\mathscr{F}$:*

(i) weak ordering

(ii) pointwise monotonicity

(iii) truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$

(iv) non-confoundness

Proof. The proof is completely analogous to the proof of the Theorem 4.13, while accounting for distinct weighting functions. □

5 Probability Measures

In this section $\mathscr{F}$ is a set of probability measures:

Axiom 5.1. [Lotteries Setup]
$\mathscr{C}$ is nonempty set of consequences, endowed with an algebra $\mathscr{D}$ containing

[19] In their result, the presence of certainty equivalent is a structural assumption, and is not necessary for the representation. In my Theorem 4.15, a weaker non-confoundness condition is both necessary and sufficient.

all singleton subsets[20]*, $\mathscr{F}$ is a set of, possibly only finitely additive, probability distributions on $\mathscr{D}$. The set $\mathscr{F}^s$ consists of simple probability measures (i.e., measures which finite support), need not be* all *simple measures. $\mathscr{F}$ is truncation-rich and extremal boundedness 2.12 holds.*

The countably additive results, the generic dominance condition $\trianglerighteq$ takes the form of stochastic dominance $\trianglerighteq_s$:

Definition 5.2. With $h, l \in \mathscr{F}$, h *stochastically dominates* ($\trianglerighteq_s$) l whenever $h\{c \in \mathscr{C} : c \preccurlyeq \alpha\} \leq l\{c \in \mathscr{C} : c \preccurlyeq \alpha\}$ for all $\alpha \in \mathscr{C}$.

For the finite additivity results, a condition stronger than stochastic dominance is needed. A counterpart of Savage's P7 for risk, as proposed by Wakker (1993)[21], is used here:

Definition 5.3. *h conditionally dominates* ($\trianglerighteq_c$) l whenever there exist $h_i, l_i \in \mathscr{F}$, $i \in 1 \ldots m$ and $\sigma_i \in [0,1]$, $\sum_{i=1}^m \sigma_i = 1$ such that $h = \sum_{i=1}^m \sigma_i h_i$, $l = \sum_{i=1}^m \sigma_i l_i$ and h_i stochastically dominates l_i for all i.

For the dominance conditions above, the definition of bounded probability measures is an immediate application of Def. 2.2 – f is *bounded from above* if there exists $\sigma \in \mathscr{C}$ such that σ stochastically dominates f, that is $f(\{c \in \mathscr{C} : c \preccurlyeq \sigma\}) = 1$. f is *bounded from below* when there exists $\sigma \in \mathscr{C}$ such that $f(\{c \in \mathscr{C} : c \succcurlyeq \sigma\}) = 1$.

Monotonicity with respect to stochastic dominance ($\trianglerighteq_s$) or conditional dominance ($\trianglerighteq_c$) is defined as in Def. 2.3. *Truncation* is as in Def. 2.5, and *truncation richness* is as in Def. 2.8:

Definition 5.4. $f^{\wedge\nu}$ is a *probabilistic truncation* of f from above, if $f^{\wedge\nu}$ is the same as f on $\{\alpha \in \mathscr{C} : \alpha \preccurlyeq \nu\} \backslash \nu$ and assigns to ν all the remaining probability $f(\nu) + f(\{\alpha : \alpha \succcurlyeq \nu\})$. Similarly, the truncation from bellow, $f_{\vee\mu}$ is the same as f on $\{\alpha \in \mathscr{C} : \alpha \succcurlyeq \mu\} \backslash \mu$ and assigns to μ all the remaining probability $f(\mu) + f(\{\alpha : \alpha \preccurlyeq \mu\})$.

A standard notation for mixtures $f_\alpha g = \alpha f + (1-\alpha) g$ is used hereafter. For later reference, two well known axioms are listed here:a

Definition 5.5. vNM-continuity holds on $\mathscr{S} \subseteq \mathscr{F}$, if $h, f, l \in \mathscr{S}$, $h \succ f \succ l$, then exists $\alpha, \beta \in (0,1)$ such that $h_\alpha l \succ f \succ h_\beta l$.

[20] As in the case of state space(4.1) this assumption is for simplicity only.

[21] The version proposed by Wakker (1993, p.473) is slightly weaker, but is considerably more complicated to state. In the presence of Wakker's simple equivalence assumption and vNM-independence on $\mathscr{F}^s$ the two conditions are equivalent. Moreover, the proofs provided here also hold if Wakker's version of conditional-monotonicity is used. Thus, our result is strictly more general.

Definition 5.6. $\succcurlyeq$ satisfies **vNM-independence** on $\mathscr{S} \subseteq \mathscr{F}$ if, for all acts $f, h, l \in \mathscr{S}$ and $\sigma \in (0,1)$,

$$h \succ l \Rightarrow h_\sigma f \succ l_\sigma f \quad .$$

5.1 Expected Utility (Neumann and Morgenstern, 1944)

The popular expected utility representation for risk of Neumann and Morgenstern (1944) has the longest history of extensions to unbounded domains. Foldes (1972); Fllmer et al. (2004); Grandmont (1972); Nielsen (1984) considered an extension of EU with an addition of various continuity assumptions. All of these are less general than the approach taken in this paper, which does not impose any topological restrictions. Fishburn (1975) and Fishburn (1982, ch.3) provided an extension of EU by imposing a conditional truncation, and his approach for countably additive measures is fully consistent with the general extension strategy pursued in this paper. For a detailed overview of the earlier work on vNM-EU extension confer Nielsen (1984). Wakker (1993) proposed an alternative to Fishburn's extension for finitely additive case that relied on conditional-monotonicity. More recently, Delbaen et al. (2011) also provided an extension of EU without requiring continuity conditions. They provided a series of extensions based on stochastic dominance. This paper generalizes all the above contributions.

The next two theorems provide representations for finitely additive and countable additive cases, and are structurally more general than the results provided before. The results of Wakker, who required simple equivalence, are straightforward corollaries and are not reproduced here. (todo: move this?) Fishburn used a weaker dominance condition, which in our setup turns out to be equivalent to stochastic dominance. He also assumed convexity of the $\mathscr{F}^s$ space, which is not required here. Instead of convexity of FS, a weaker notion of fine-rangeness of $\mathscr{F}^s$ is used.

Definition 5.7. [Fine-rangeness of $\mathscr{F}^s$] For any $m, M \in \mathscr{C}$, $m \prec M$, and $p \in [0,1], \epsilon > 0$ there exist a simple probability $m_\pi M \in \mathscr{F}^s$ such that $\pi \in (p-\epsilon, p+\epsilon)$.

Fine rangeness is immediate when binary simple equivalence holds, that is for every lottery f there exist a binary lottery $m_p M \sim f$. Virtually all results in the risk literature imply binary simple equivalence as a consequence of convexity of $\mathscr{F}$ and vNM-continuity or a similar condition. For example if EU holds on $\mathscr{F}^s$ and $\mathscr{F}^s$ is convex, then simple equivalence, and thus fine-rangeness are immediate. Hence, fine rangeness does not impose additional constraints.

Theorem 5.8. *Under the lotteries setup (5.1), assume that EU holds on $\mathscr{F}^s$ with $\mathscr{F}^s$ finely-ranged. An unique EU holds on the whole set $\mathscr{F}$ with the same utility $U : \mathscr{C} \to \mathbb{R}$, if and only if the following conditions hold:*

(i) weak order

(ii) conditional-monotonicity

(iii) truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$

(iv) non-confoundness

Proof of theorem 5.8. First, I show that simple denseness holds. Let $EU(l^s) < EU(h^s)$ for $f^s, h^s \in \mathscr{F}^s$. Take the outcomes $m, M \in \mathscr{C}$ to be common lower an upper bounds of l^s and h^s. By fine-rangeness of P, for all $\epsilon \in (EU(l^b), EU(h^b))$, hence, there must exist π_ϵ^- and π_ϵ^+, such that $EU(l^s) < EU(m_{\pi_\epsilon^-}M) < \epsilon < EU(m_{\pi_\epsilon^+}M) < EU(h^s)$, and simple denseness holds.

For *necessity*, weak ordering is obvious and conditional monotonicity follows immediately because of the linearity of the integral with respect to mixtures. Truncation approximations also follows from the monotone convergence of the Lebesgue integrals theorem. For non-confoundness, let $h, l \in \mathscr{F}^b$ such that $EU(h) > EU(l)$. Take $M, m \in \mathscr{C}$ such that $U(M) \geq EU(h) > EU(l) \geq U(m)$. By fine-rangeness, similarly to the proof of simple denseness above, non-confoundness follows on $\mathscr{F}^b$. For $h, l \in \mathscr{F} \backslash \mathscr{F}^b$, by truncation richness and truncation monotonicity there exist close enough truncated (from both sides if needed) acts $b^h, b^l \in \mathscr{F}^b$ such that $EU(b^h) > EU(b^l)$, $EU(b^h) > EU(l)$ and $EU(h) > EU(b^l)$, and we are in previous case of bounded h, l.

For *sufficiency* all the conditions of the Theorem 3.5, except simple approximation on $\mathscr{F}^b$, are satisfied. Simple approximation follows from the following considerations. Let $m, M \in \mathscr{C}$ the bounds of bounded lottery P. Cut the interval $[U(m), U(M)]$ in n sub-intervals $A_1 = [U(m) = u_0, u_1], A_2 = (u_1, u_2], \ldots, A_n = (u_{n-1}, u_n = U(M)]$. Let $\Delta_i = U^{-1}(A_i)$. By fine-rangeness, for all i, such that $P(\Delta_i) > 0$, there exists a simple density $m_{\sigma_i}M$ such that $\sigma_i \in (P(\Delta_i) - \epsilon, P(\Delta_i))$. Obviously, the conditional density P_{Δ_i} stochastically dominates $m_{\sigma_i}M$ for all i, $P(\Delta_i) > 0$. Define $P_{\epsilon,n}^- = \sum_{i=1}^n \frac{\sigma_i}{\Sigma}(m_{\sigma_i}M)$, were $\Sigma = \sum_{i=1}^n \sigma_i$ and note that $P_{\epsilon,n}^-$ might not be in $\mathscr{F}^s$, but by fine rangeness there exists $\tilde{P}_{\epsilon,n}^- \in \mathscr{F}^s$, arbitrary close to $P_{\epsilon,n}^-$ and dominated stochastically by $P_{\epsilon,n}^-$. By conditional dominance $P \succcurlyeq P_{\epsilon,n}^-$ for all ϵ and n. Similarly construct $P_{\epsilon,n}^+$ conditionally dominating P. Then $EU(\tilde{P}_{\epsilon,n}^-) - EU(\tilde{P}_{\epsilon,n}^+) \to 0$ and by Lemma 2.10 simple approximation on $\mathscr{F}^b$ holds.

Thus, by theorem 3.5 there exist an unique F^{**} defined as in Eq. (3.3). By sufficiency, EU satisfies all the conditions of the theorem and thus $F^{**} = EU$ on $\mathscr{F}$ and is finitely valued.

$\square$

The countably additive case is considerably easier to deal with. In the presence of countable additivity, stochastic dominance is enough to obtain the extension on the whole $\mathscr{F}$.

Theorem 5.9. *Under the lotteries setup (5.1), assume that $\mathscr{F}$ is a set of countably additive probability measures with $\mathscr{F}^s$ finely-ranged, and EU holds on $\mathscr{F}^s$. Then an unique EU holds on the whole $\mathscr{F}$ with the same utility $U : \mathscr{C} \to \mathbb{R}$, if and only if the following conditions hold:*

(i) weak order

(ii) stochastic dominance on $\mathscr{F}$

(iii) truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$

(iv) non-confoundness

Moreover, if Axiom 2.12 holds, or $U(\mathscr{C})$ is bounded in IR, then $|EU(f)| < \infty$ for all $f \in \mathscr{F}$.

Proof of theorem 5.9. Proofs of simple denseness, necessity, uniqueness and boundedness in value, are the same as those in the proof of Theorem 5.8.

It remains to prove simple approximation. Let $m, M \in \mathscr{C}$ the bounds of bounded lottery P. Without loss of generality, rescale U(m)=0, U(M)=1. Cut the interval $[U(m), U(M)]$ in n sub-intervals $A_1 = [U(m), \frac{1}{n}], A_2 = (\frac{1}{n}, \frac{2}{n}], \ldots, A_n = (\frac{n-1}{n}, U(M)]$. Let $\Delta_i = U^{-1}(A_i)$.

Construct a simple cumulative probability distribution s^- as follows. Let $u_i = \inf\{U(f(c)) : c \in \Delta_i\}$. If there exists $c_i \in \Delta_i$ such that $U(c_i) = u_i$ then define $s^-(c_i) = P(\{x : U(x) \leq u_i\})$, and set $B_i = \emptyset$. If not, take decreasing sequence of intervals $E_j = (u_i, \epsilon_j]$, $E_j \to \emptyset$. By countable additivity $P(U^{-1}(E_j)) \to \emptyset$, and there must exist E_k such that $P(U^{-1}(E_k)) < \frac{1}{n^{i+1}(i+1)}$. Take arbitrary $c_i \in U^{-1}(E_k)$ and set $s^-(c_i) = P(\{x : U(x) \leq u(c_i)\})$. Define $B_i = (u_i, U(c_i)]$. The simple probability measure s^-, constructed in this way, is arbitrary close in EU value to P and $P \trianglerighteq_s s^-$. To see this, denote F_μ, a distribution function generated by measure μ and utility U, i.e.,

$F_\mu(x) = P(c : U(c) \leq x)$, then

$$EU(P) - EU(s^-) = \tag{5.1}$$

$$\sum_{i=0}^{n-1} \int_{A_i \setminus B_i} \left[F_{s_i^-} - F_P(x) \right] dx + \sum_{i=0}^{n-1} \int_{B_i} \left[F_{s_i^-} - F_P(x) \right] dx < \tag{5.2}$$

$$\frac{1}{n} + \sum_{i=1}^{n} \frac{i}{n^i i} < \frac{1}{n} + \frac{1}{n-1} < \frac{2}{n-1} \tag{5.3}$$

Similarly we can find a simple s^+ such that $s^+ \trianglerighteq_p f$ and $EU(s^+) - EU(f) < \frac{2}{n-1}$. Thus $EU(s^+) - EU(s^-) < \frac{4}{n-1}$, and by lemma 2.10 simple approximation on $\mathscr{F}^b$ holds.

□

Fishburn (1975) used condition ($P5^*$), which can be called *conditional truncation approximation*, because it is based on the following definition of truncations:

Definition 5.10. $f^{\wedge\nu}$ is a *conditional truncation* of f from above, if $f^{\wedge\nu}$ is the conditional distribution of f on $\{\alpha \in \mathscr{C} : \alpha \preccurlyeq \nu\}$. Similarly, the truncation from bellow $f_{\vee\mu}$ is the conditional distribution of f on $\{\alpha \in \mathscr{C} : \alpha \succcurlyeq \mu\}$.

Obviously, in order for the above definition to be useful, a version of truncation richness must be assumed. Fishburn assumed that all the conditional probability distributions are in the domain. He also assumed that $\mathscr{F}$ is convex, and all degenerate distributions are in the domain. When EU is given on $\mathscr{F}^s$, convexity implies simple equivalence, and thus also implies simple denseness and fine rangeness. Fishburn's dominance condition is weaker than (but equivalent in the presence of other conditions) stochastic dominance 5.2. The next theorem is a version of Fishburn's theorem.

Theorem 5.11. *Under the lotteries setup (5.1), with $\mathscr{F}$ a set of countably additive probability measures, $\mathscr{F}^s$ is finely-ranged, and EU holds on $\mathscr{F}^s$. Then an unique EU holds on the whole $\mathscr{F}$ with the same utility $U : \mathscr{C} \to \mathbb{R}$, if and only if the following conditions hold:*

(i) weak order

(ii) stochastic dominance on $\mathscr{F}$

(iii) conditional truncation approximation on $\mathscr{F} \setminus \mathscr{F}^b$

(iv) non-confoundness

Moreover, if Axiom 2.12 holds, or U is bounded in $\mathbb{R}$, then $|EU(f)| < \infty$ for all $f \in \mathscr{F}$.

Proof. With the slight modifications accounting for a different version of truncation, the proof is identical to the proof of the Theorem 5.9 □

5.2 Betweenness Model (Fishburn, 1983)

One of the most notable relaxations of the vNM-independence axiom was achieved by a series of *betweenness models* (Chew and MacCrimmon, 1979; Chew, 1983; Dekel, 1986; Fishburn, 1983). Any functional $V(\cdot)$ on the set of lotteries $\mathscr{F}$ that satisfies $\forall h, l \in \mathscr{F}, \forall \lambda \in [0, 1]$,

$$u = V(h) = V(l) \Rightarrow V(h_\lambda l) = u$$

is a *betweenness functional.* In other words, sets on which the betweenness functional is constant are convex.

The following extension is based on the axiomatization of Fishburn (1983). The one of Dekel (1986) is similar, but uses more restrictive boundedness condition, which made it possible to derive an implicit representation of $V(\cdot)$ as a solution of an integral equation close in form to the EU functional. An important particular case, with an explicit integral form is *weighted utility* axiomatized by Chew and MacCrimmon (1979) and Chew (1983) among others.

It is well known that the only non-technical condition that is necessary for the existence of the betweenness representation V is mixture-betweenness:

Definition 5.12. Mixture-betweenness holds if $\forall h, l \in \mathscr{F}$, with $h \succ l$ we have

$$h \succ h_\lambda l \succ l \quad .$$

First of all, we need to define V for unbounded lotteries:

Definition 5.13. An **extended betweenness functional** is a betweenness functional V that satisfies the following properties:

$$V = \begin{cases} \sup_\nu V(f^{\wedge\nu}) \text{ for } f \text{ bounded below} \\ \inf_\mu V(f_{\vee\mu}) \text{ for } f \text{ bounded above} \\ \sup_\nu \inf_\mu V(f^{\wedge\nu}_{\vee\mu}) \text{ for } f \text{ unbounded from both sides} \end{cases} .$$

Fishburn (1983) required $\mathscr{F}^b$ to be *countably-bounded* – there exists a countable subset $\mathscr{B}$ of $\mathscr{F}^b$ such that for every $f \in \mathscr{F}^b$ there exists $h, l \in \mathscr{B}$

such that $l \preccurlyeq f \preccurlyeq h$. For example, this condition is satisfied when $\mathscr{C}$ is a convex subset of $\mathbb{R}^n$, $\mathbb{Q}^\ltimes$ or $\mathbb{N}^n$.

Fishburn (1983) have already proved the betweenness representation on the set of bounded acts $\mathscr{F}^b$. The next theorem extended the representation from $\mathscr{F}^b$ to $\mathscr{F}$, by means of the Theorem 3.3.

Theorem 5.14. *Under the lotteries setup (5.1), and countably bounded and convex $\mathscr{F}^b$, an extended betweenness functional, V, represents $\succcurlyeq$ on $\mathscr{F}$, if and only if the following conditions hold*

(i) weak ordering

(ii) vNM-continuity on $\mathscr{F}$

(iii) mixture-betweenness on $\mathscr{F}^b$

(iv) stochastic dominance

(v) truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$

Moreover, the functional V is monotonic with respect to stochastic dominance, and $V(h_\lambda l)$ is continuous and increasing in $\lambda \in [0,1]$ for $h \succ l$.

Proof. Fishburn (1983, Theorem 1) proved that conditions (i)-(iii) imply the existence of the betweenness functional V on the set of countably bounded $\mathscr{F}$. By vNM-continuity, non-confoundness easily follows, as every f can be enclosed by a pair of bounded lotteries. By the extension theorem 3.3, the extended betweenness representation V exists and has all the properties as required by the theorem. □

5.3 Disappointment Aversion (Gul, 1991)

Gul (1991) proposed an appealing model for decision under risk, which incorporates a natural idea that outcomes higher than a certainty equivalent have an elation valence, and the lower than certainty equivalent outcomes are interpreted as disappointment ones. Gul's model is a betweennes model, but it has a close resemblance with the EU functional and is similarly tractable. Formally, a certainty equivalent C_P of a lottery f in the disappointment aversion model is a solution to the following implicit equation:

$$DA(f) := U(C_f) = (1 - \gamma(\alpha_f))EU(f_\prec) + \gamma(\alpha_f)EU(f_\succcurlyeq) \tag{5.4}$$

where $A_\prec = \{c : c \prec C_f\}$, $A_\succcurlyeq = \{c : c \succcurlyeq C_f\}$ and $\alpha_f = f(\{c : c \succcurlyeq C_f\})$ is a probability to receive higher outcome than the certainty equivalent of

f, called an *elation* probability. $f_{\prec}, f_{\succcurlyeq}$ are the conditional f lotteries given non-empty sets $A_{\prec}$ and $A_{\succcurlyeq}$. The weighting function $\gamma : [0,1] \to [0,1]$ is:

$$\gamma(\alpha) = \frac{\alpha}{1 + (1-\alpha)\beta} \tag{5.5}$$

If $\beta > 0$ then γ is strictly convex, and thus $\gamma(\alpha) < \alpha$, which means that the part of f lower than C_f is overweighted, and the part or f higher than C_f is underweighted, leading to disappointment aversion.

With the γ as in (5.5), Eq. (5.4) can be written in an alternative form[22]:

$$U(C_f) = (1-\alpha_f)(1+\beta)EU(f_{\prec}) + \gamma(\alpha_f)EU(f_{\succcurlyeq}) - (1-\alpha_f)\beta U(C_f) = \tag{5.7}$$

$$= EU(f) - \beta \left[\int_{x \prec C_f} (U(C_f) - U(x))df \right] \tag{5.8}$$

With the notation

$$DA(f, C) := EU(f) - \beta \left[\int_{x \prec C} (U(C) - U(x))df \right] \tag{5.9}$$

the DA model is given by the solution of the implicit equation $DA(f, C) = U(C)$.

With an increasing utility U the DA satisfies stochastic dominance. To see this, assume for contradiction that $H \trianglerighteq_s L$, but $U(C_H) < U(C_L)$. Second part of the Eq. (5.9) is positive and increasing in $U(C)$, and by stochastic dominance,

$$DA(H, C) \geq DA(L, C)$$

as for every $C \in \mathscr{C}$. Both sides of the above inequality are decreasing in $U(C)$, and thus $U(C_H) = DA(H, C_H) \geq DA(H, C_L) \geq DA(L, C_L) = U(C_L)$ and a contradiction has resulted.

Gul (1991) provided his model on the set of simple lotteries taking values into a bounded closed interval in $\mathbb{R}$. The definitions and theorems, proposed in this paper, apply to a convex set of lotteries $\mathscr{F}$ defined on a general outcome space $\mathscr{C}$. There is no emphasis on fine ranginess here, as the very definition of DA requires the existence of the certainty equivalent.

[22] This formulation was used by ROUTLEDGE and ZIN (2010) to state their *generalized disappointment aversion* model:

$$U(C_f) = EU(f) - \beta \left[\int_{x \prec \delta C_f} (\delta U(C_f) - U(x))df \right] \tag{5.6}$$

with an additional parameter $\delta > 0$, and real outcomes $\mathscr{C} = \mathbb{R}^+$.

Theorem 5.15. *Under the lotteries setup (5.1), with $\mathscr{F}$ a set of countably additive lotteries, assume that DA holds on $\mathscr{F}^s$ with utility $U : \mathscr{C} \to \mathbb{R}$ and $\mathscr{F}^s$ is convex. Then an unique DA holds on the whole $\mathscr{F}$, if and only if the following conditions hold:*

(i) weak order

(ii) stochastic dominance on $\mathscr{F}$,

(iii) truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$

(iv) non-confoundness

Moreover, if Axiom 2.12 holds, or $U(\mathscr{C})$ is bounded in $\mathbb{R}$ then $DA(\cdot)$ is finite-valued.

Proof of theorem 5.15. By convexity of $\mathscr{F}^s$, simple denseness holds. Stochastic dominance was proved in the main text. Truncation approximation can be proved from the equation (5.7) (todo: really?). Non-confoundness is implied by the existence of the certainty equivalents.

Simple approximation is proved as follows. As was shown in the proof of theorem 5.9, for any bounded lottery f it is possible to find a sequence of simple functions s_n^- and s_n^+ converging in EU value to f and $s^+ \trianglerighteq_s f \trianglerighteq_s s^-$. Let $U_H := \lim_{n\to\infty} U(C_{s_n^+})$ and $U_L := \lim_{n\to\infty} U(C_{s_n^-})$. By stochastic dominance $U_H \geq U_L$. Assume for contradiction that $U_H > U_L$, then:

$$U_H - U_L = \lim_{n\to\infty} [DA(s_n^+) - DA(s_n^-)] = \tag{5.10}$$

$$\lim_{n\to\infty} \left[EU(s_n^+) - EU(s_n^-)\right] + \tag{5.11}$$

$$\lim_{n\to\infty} \left[\int_{x \prec C_{s_n^-}} (U(C_{s_n^-}) - U(x))ds_n^- - \int_{x \prec C_{s_n^+}} (U(C_{s_n^+}) - U(x))ds_n^+\right] = \tag{5.12}$$

$$0 + \lim_{n\to\infty} \left[\int_{x \prec C_{s_n^-}} U(C_{s_n^-})ds_n^- - \int_{x \prec C_{s_n^-}} U(C_{s_n^+})ds_n^+\right] \tag{5.13}$$

$$-\lim_{n\to\infty} \int_{C_{s_n^-} \preccurlyeq x \prec C_{s_n^+}} (U(C_{s_n^+}) - U(x))ds_n^+ = \tag{5.14}$$

$$f(\{x : U(x) \prec U_l\})(U_L - U_H) - \lim_{n\to\infty} \int_{C_{s_n^-} \preccurlyeq x \prec C_{s_n^+}} (U(C_{s_n^+}) - U(x))ds_n^+ \leq 0 \tag{5.15}$$

a contradiction has resulted. Thus $U_H = U_L$ and by lemma 2.10 simple approximation on $\mathscr{F}^b$ follows. Note what countable additivity was used for the transition in the last equality in the derivation above.

Thus, by theorem 3.5, there exist an unique F^{**} defined as in Eq. (3.3). By sufficiency, $DA = F^{**}$ which proves the theorem. □

A full statement of the extended Gul's representation follows.

Corollary 5.16. [Gul (1991) *DA* extension] *Under lottery setup 5.1, $\mathscr{C} = \mathbb{R}$ and $\mathscr{F}$ a convex set of countably additive lotteries on $\mathbb{R}$, DA holds if and only if*

(i) weak ordering

(ii) vNM-continuity on $\mathscr{F}$

(iii) Gul (1991) weak independence on $\mathscr{F}^s$

(iv) Gul (1991) symmetry on $\mathscr{F}^s$

(v) stochastic dominance on $\mathscr{F}$

(vi) truncation approximation

Conditions *(i)-(iv)* are the original Gul's axioms.

6 State-lotteries

Because of its analytical tractability, the Anscombe and Aumann (1963) framework served as a basis for numerous decision models. The most famous are *EU* of Anscombe and Aumann (1963), Choquet *EU* of Schmeidler (1989) and MaxMin *EU* of Gilboa and Schmeidler (1989). In this subsection I generalize these results to unbounded acts.

Wakker (1993) already gave an extended representation for Anscombe and Aumann and Schmeidler models. The results provided here, being based on the main Theorem 3.5, are structurally and logically more general than those of Wakker. The extension of Gilboa and Schmeidler (1989) MaxMin model is provided for the first time in the literature.

We formalize the Anscombe and Aumann (1963) framework in the following structural assumption:

Axiom 6.1. [State-lotteries assumption] *Assume the state space (4.1) setup with $\mathscr{C}$ a convex set of simple lotteries over a set Γ. Acts $f, g \in \mathscr{F}$ can be mixed in a pointwise manner:*

$$f_\alpha g : \omega \to f(\omega)_\alpha g(\omega)$$

6.1 Expected Utility (Anscombe and Aumann, 1963) and Choquet Expected Utility (Schmeidler, 1989)

Acts $f, g \in \mathscr{F}$ are *comonotonic* if for all $w_1, w_2 \in \Omega$

$$f(w_1) \succcurlyeq f(w_2) \Leftrightarrow g(w_1) \succcurlyeq g(w_2) \quad .$$

The relation $\succcurlyeq$ is said to satisfy *comonotonic vNM-independence* condition if f, h, l in the definition of vNM-independence (5.6) are pairwise comonotonic.

The definition of CEU for state-lotteries framework is virtually the same as the one given for state space setup in section 4.2, with the difference that U in replaced by a linear in mixtures EU functional: $CEU(f) = \int_\Omega EU(f(\omega))dv(\omega)$. With $\trianglerighteq$ as pointwise dominance $\trianglerighteq_p$ (Def. 4.5) and monotonicity as pointwise monotonicity (Def. 4.6) the CEU representation becomes:

Theorem 6.2. Extension of Schmeidler (1989) CEU *Under state-lotteries (6.1) assumptions with convex $\mathscr{F}^s$, $\succcurlyeq$ maximizes CEU with respect to a capacity ν and a function $EU : \mathscr{C} \to \mathbb{R}$ defined by a monotonic function $U : \Gamma \to \mathbb{R}$, if and only if the following conditions are satisfied:*

(i) *weak ordering*

(ii) *vNM-continuity on $\mathscr{F}^s$*

(iii) *comonotonic vNM-independence on $\mathscr{F}^s$*

(iv) *pointwise monotonicity*

(v) *truncation approximation on $\mathscr{F} \backslash \mathscr{F}^b$*

(vi) *non-confoundness*

Proof. Schmeidler (1989) proved that (i)-(iii) are necessary and sufficient for the representation on $\mathscr{F}^s$. The rest of the proof is identical to the proof of theorem 4.13. □

As usual, EU representation follows immediately by strengthening the independence condition:

Corollary 6.3. [Expected Utility of Anscombe and Aumann (1963)] *In the above theorem $\succcurlyeq$ maximizes EU if and only if comonotonic vNM-independence is strengthened to vNM-independence.*

6.2 Max-Min Expected Utility (Gilboa and Schmeidler, 1989)

Gilboa and Schmeidler (1989) provided an axiomatization of MaxMin Expected Utility. The first mention of this model is due to Wald (1949) and has a long history in statistics. With the notation and framework as defined in the state-lotteries setup (6.1) this model is defined as:

Definition 6.4. Max-Min Expected Utility (MMEU) holds on $\mathscr{F}$ if there exist an utility function $U : \Gamma \to \mathbb{R}$ and a non-empty, closed and convex set Π of probability measures on $\mathscr{A}_\Omega$ such that

$$MMEU(f) = \min_{f \in \Pi} \int EU(f(w))df(w)$$

represents $\succcurlyeq$ on $\mathscr{F}$.

Gilboa and Schmeidler (1989) used two new conditions, *certainty-vNM-independence* – vNM-independence imposed only on certain prospects, i.e., elements of $\mathscr{C}$, and *uncertainty-aversion* – for all $f, g \in \mathscr{F}$, $f \sim g$ and $\alpha \in (0,1)$ implies $\alpha f + (1-\alpha)g \succcurlyeq f$.

Theorem 6.5. **[Extension of Gilboa and Schmeidler (1989) MMEU]** *Under the state-lotteries (6.1) and Axiom 2.12 assumptions, $\succcurlyeq$ maximizes finite-valued $MMEU$ with respect to set of probability measures Π and a function $EU : \mathscr{C} \to \mathbb{R}$ defined by a monotonic function $U : \Gamma \to \mathbb{R}$, if and only if the following conditions are satisfied:*

(i) weak ordering

(ii) vNM-continuity on $\mathscr{F}^s$

(iii) certainty-vNM-independence on $\mathscr{F}^s$

(iv) uncertainty-aversion on $\mathscr{F}^s$

(v) pointwise monotonicity on $\mathscr{F}$

(vi) truncation approximation on $\mathscr{F} \setminus \mathscr{F}^b$

(vii) non-confoundness

Proof. For sufficiency, truncation approximation and non-confoundness follow as in the proof of Theorem 4.9. Gilboa and Schmeidler (1989) proved that (i)-(v) are necessary and sufficient for the representation to hold on $\mathscr{F}^s$.

The rest of the proof is based on Theorem 3.5. Simple denseness is implied by convexity of FS^s. Thus all the conditions of theorem 3.5 except simple approximation are immediately satisfied.

It remains to prove simple approximation. Let $f \in \mathscr{F}^b$. Following the same constructive argument of the proof of Theorem 4.10, we can find a sequence of simple acts s_n^- pointwise dominated by f, and simple acts s_n^+ pointwise dominating f, such that $\int EU(s_n^+(w))df(w) - \int EU(s_n^-(w))df(w) \leq \frac{1}{n}$, for all $\pi \in \Pi$. Thus, $MMEU(s_n^+) - MMEU(s_n^-) \leq \frac{1}{n}$, and by Lemma 2.10, simple approximation holds.

By theorem 3.5 there exist an unique extension F^{**} of $MMEU$ on $\mathscr{F}$, and F^{**} is finite-valued. By sufficiency argument, $F^{**} = MMEU$.

3.5 $MMEU = F^{**}$ on $\mathscr{F}$. □

7 Conclusions

This paper has provided a general strategy for extensions of behavioral foundations. Given a representation on a set of simple objects, monotonicity, truncation approximation and non-confoundness are usually enough to assure the existence of a unique and well defined extension to the whole domain of objects of interest. Behavioral and decision scientists from a diverse spectrum of fields, be it inter-temporal choice, uncertainty, welfare, or multi-criteria aggregation, can apply the proposed strategy to easily extend the existing representations.

Many examples from decision under risk and uncertainty were given. Extension theorems in this paper are more general, both in a structural and logical sense, than those that have been provided before. For the first time several well known models are fully extended: Fishburn (1983) betweennes model, Gul (1991) disappointment aversion model and MaxMin expected utility of Gilboa and Schmeidler (1989).

A Weak non-confoundness

As mentioned in the main text, a strong version of non-confoundness as defined in the Def. 2.9 can be relaxed in the presence of simple approximation and truncation approximation. This appendix clarifies this statement.

Non-confoundness is trivially satisfied on $\mathscr{F}^s$ and immediately follows from simple approximation on $\mathscr{F}^b$. In presence of truncation approximation, non-confoundness is implied only partially on $\mathscr{F} \backslash \mathscr{F}^b$, and confoundness can still occur for objects $h \succ l$, where h is unbounded below and l is unbounded above. The following condition would exclude such an anomalous case.

Definition A.1. *Weak non-confoundness* holds on $\mathscr{F}$ if for every $h, l \in \mathscr{F}$, $h \succ l$, there exists simple objects $b_h, b_l \in \mathscr{F}^b$

$$h \succcurlyeq b_h \succ l \qquad \text{whenever } h \text{ is unbounded below} \tag{A.1}$$

$$h \succ b_l \succcurlyeq l \qquad \text{whenever } l \text{ is unbounded above} \tag{A.2}$$

Lemma A.2. *If simple-approximation holds on $\mathscr{F}^b$, truncation approximation holds on $\mathscr{F} \backslash \mathscr{F}^b$, and weak non-confoundness holds, then non-confoundness holds on $\mathscr{F}$.*

Proof. Let $h \succ l$. Assume, first, that $h \in \mathscr{F}^b$. There are several cases to consider:

(i) $l \in \mathscr{F}^b$
By simple approximation non-confoundness holds.

(ii) l is unbounded above
By weak non-confoundness and simple approximation, non-confoundness holds.

(iii) l is unbounded below and bounded above
By truncation approximation there exists $l_{\vee\mu}$ such that $h \succ l_{\vee\mu} \succcurlyeq l$ and we are in the case (i) from above.

If h is bounded below and unbounded above, by truncation approximation, there exists $h^{\wedge\nu}$ such that $h \succcurlyeq h^{\wedge\nu} \succ l$ and by cases above, non-confoundness holds. If h is unbounded below, by weak non-confoundness, there is b such that $h^{\wedge\nu} \succcurlyeq b \succ l$ and we are in one of above cases again. □

Thus, in all main results in the section 3 non-confoundness can be replaced by weak non-confoundness assumption.

B Wakker's (1993) representation of EU

In this section I give a complete extension of *EU* representation based on Wakker's (1993) conditional monotonicity and sure thing principle conditions.

Definition B.1. *Weak conditional dominance* $\trianglerighteq_c$ for $h, l \in \mathscr{F}$ and $s \in \mathscr{F}^s$ is defined as follows:

- $h \trianglerighteq_c s$ if there exists a finite partition of Ω, $\{A_1, \ldots A_n\}$ such that $(\forall A_i, \forall \omega \in A_i : h(\omega)_{A_i} \succcurlyeq s_{A_i})$
- $s \trianglerighteq_c l$ if there exists a finite partition of Ω, $\{A_1, \ldots A_n\}$ such that $(\forall A_i, \forall \omega \in A_i : s_{A_i} \succcurlyeq l(\omega)_{A_i})$.

This leads to the weak form of Savage's $P7$ dominance condition:

Definition B.2. $\succcurlyeq$ satisfies *weak conditional monotonicity* on $\mathscr{F}$ if for all $h \in \mathscr{F}$ and $s \in \mathscr{F}^s$

$$\begin{aligned} h \trianglerighteq_c s &\Rightarrow h \succcurlyeq s \\ s \trianglerighteq_c h &\Rightarrow s \succcurlyeq h \quad . \end{aligned}$$

At the expense of being slightly more elaborate than Savage's $P7$ axiom[23], the above condition is weaker. To be unambiguously defined, this version of conditional-monotonicity requires that the sure-thing principle to hold only on $\mathscr{F}^s$. Savage's version requires sure-thing principle to hold on the whole set $\mathscr{F}$.

Theorem B.3. [Savage EU with finitely additive probability] *Assume that state space setup 4.1 and Axiom 2.12 hold and $\mathscr{A}_\Omega$ is an algebra. Finitely valued EU represents $\succcurlyeq$ on $\mathscr{F}$ with respect to a finely ranged probability measure P and utility U, if and only if the following conditions hold:*

P1 *weak ordering*

P2 *sure-thing principle on $\mathscr{F}^s$ on $\mathscr{F}^s$*

P3 *if A is non-null then $\forall \alpha, \beta \in \mathscr{C} : \alpha \succcurlyeq \beta \Leftrightarrow \alpha_A \succcurlyeq \beta_A$*

P4 *if $\alpha \succ \beta$ and $\gamma \succ \delta$, $\alpha, \beta, \gamma, \delta \in \mathscr{C}$ then for events A,B: $[\alpha_A \beta \succcurlyeq \alpha_B \beta] \Leftrightarrow [\gamma_A \delta \succcurlyeq \gamma_B \delta]$*

P5 *$x \succ y$ for some $x, y \in \mathscr{F}$*

P6 *if for $f, h \in \mathscr{F}^s$, $f \succ h$ and $\alpha \in \mathscr{C}$, then there exists a partition $(A_1, \cdots, A_m)$ of Ω, $A_i \in \mathscr{A}_\Omega$, such that $\alpha_{A_i} f \succ h$ and $f \succ \alpha_{A_i} h$ for all i*

P7 *weak conditional monotonicity*

[23] In Savage's condition, s is allowed to be from $\mathscr{F} \backslash \mathscr{F}^s$.

P'8 *truncation approximation on* $\mathscr{F} \backslash \mathscr{F}^b$

P'9 *non-confoundness*[24]

Proof. Kopylov (2007) proved that conditions P1-P6 are necessary and sufficient for the SEU representation to hold on the set of simple acts $\mathscr{F}^s$ with finely-ranged subjective probability P and non-constant U. By Theorem 4.9, EU holds on the whole $\mathscr{F}$ and is finite-valued. □

C Proofs for Sections 3

Proof of Lemma 2.10. By assumption of the lemma, for each $h \succ l$, $h, l \in \mathscr{S}$, and every $\epsilon \in \mathbb{R}^+$ there must exist $s_h^+ \trianglerighteq h \trianglerighteq s_h^-$, $s_l^+ \trianglerighteq l \trianglerighteq s_l^-$ such that $F(s_h^+) - F(s_h^-) < \epsilon$ and $F(s_l^+) - F(s_l^-) < \epsilon$.

By non-confoundness there exists s_h such that $h \succcurlyeq s_h \succ l$. By another application of non-confoundness there exists s_l such that $h \succcurlyeq s_h \succ s_l \succcurlyeq l$. By assumption, for $\epsilon^* < F(s_h) - F(s_l)$ there exists s_l^+ such that $F(s_h) > F(s_l^+)$. Similarly for $\epsilon^{**} < F(s_h) - F(s_l^+)$ there exists $F(s_h^-)$ such that $F(s_h^-) > F(s_l^+)$. As F represents $\succcurlyeq$, $h \succcurlyeq s_h^- \succ s_l^+ \succcurlyeq l$, which proves simple approximation. □

Proof of Lemma 3.4. Let $f \in \mathscr{F}$ bounded bellow (case of bounded above f is analogous). Assume for contradiction that $f^{\wedge\nu} \succ f$. Then there exists $b \in \mathscr{F}^b$ such that $f^{\wedge\nu} \succ b \succcurlyeq f$ (by non-confoundness if f is unbounded above; take $b = f$ otherwise). By simple approximation there exists s such that $f^{\wedge\nu} \succcurlyeq s \succ b \succcurlyeq f$ and $f^{\wedge\nu} \trianglerighteq s$. By the definition of truncation, $f \trianglerighteq s$, and by $\trianglerighteq$-monotonicity $f \succcurlyeq s$, a contradiction. Thus truncation monotonicity holds for any f bounded from one side.

Let f be unbounded from both sides. Assume for contradiction $f^{\wedge\nu} \succ f$. By truncation approximation there exists μ such that $f^{\wedge\nu} \succ f^{\wedge\nu}_{\vee\mu} \succcurlyeq f$. But by previous paragraph, $f^{\wedge\nu} \preccurlyeq f^{\wedge\nu}_{\vee\mu}$, a contradiction. □

Proof of Theorem 3.1. By assumption of the theorem, $F = F^*$ on $\mathscr{F}^s$. Let $f \in \mathscr{F}^b$. By the definition of the bounded objects, the sets $\{l^s : s \trianglelefteq f\}$ and $\{h^s : h^s \trianglerighteq f\}$ are non-empty. Assume for contradiction $\sup_{l^s \trianglelefteq f} F(l^s) < \inf_{h^s \trianglerighteq f} F(h^s)$. By simple deneness there must exists $s \in \mathscr{F}^s$ such that $\sup_{f \trianglerighteq l^s} F(l^s) < F(s) < \inf_{h^s \trianglerighteq f} F(h^s)$, and $f \nsim s$ (if $f \sim s$, apply simple

[24] As a side note, non-confoundness would not be necessary here if we would have imposed sure-thing principle on the whole $\mathscr{F}$ as Savage did, and would explicitly impose simple approximation. Then non-confoundness would easily follow and our main Theorem 3.5 could be directly used.

denseness again). If $f \prec s$, by simple approximation there exists a simple object f^s such that $f \preccurlyeq f^s \prec s$ and $f \trianglelefteq f^s$. Given that F represents $\succcurlyeq$ on $\mathscr{F}^s$ and monotonicity, the contradiction $F(f^s) < F(s) < \inf_{h^s \trianglerighteq f} F(h^s)$ follows. If $f \succ s$, a symmetric argument yields the desired contradiction.

Define $F^*(f) = \sup_{f \trianglerighteq l^s}(F(l^s)) = \inf_{h^s \trianglerighteq f}(F(h^s))$, $l^s, h^s \in \mathscr{F}^s$. Next is to prove that F^* indeed represents $\succcurlyeq$ on $\mathscr{F}^b$.

Assume for contradiction $h \succcurlyeq l$ and $F^*(h) < F^*(l)$ for some $h, l \in \mathscr{F}^b$. By the definition of F^* there must exist h^s, l^s, $h \trianglelefteq h^s, l^s \trianglelefteq l$ such that $F(h^s) < F(l^s)$. F^* represents $\succcurlyeq$ on $\mathscr{F}^s$ and by monotonicity $h \preccurlyeq h^s \prec l^s \preccurlyeq l$, which by transitivity contradicts the assumption. Hence

$$h \succcurlyeq l \Rightarrow F^*(h) \geq F^*(l) \quad .$$

It remains to prove that if $h \succ l$ then $F^*(h) \neq F^*(l)$. Assume for contradiction $h \succ l$ and $F^*(h) = F^*(l)$. By double application of simple approximation there exist $h^s, l^s \in \mathscr{F}^s$ such that $h \succcurlyeq h^s \succ l^s \succcurlyeq l$. By previous paragraph $F^*(h) \geq F^*(h^s) > F^*(l^s) \geq F^*(l)$ which proofs the result.

To prove uniqueness, assume that G^{**} also represents $\trianglerighteq$ on $\mathscr{F}^b$, but $F^{**}(f) > G^{**}(f)$ for some $f \in \mathscr{F}^b \backslash \mathscr{F}^s$. By definition of F^{**} there must exist $f^s \in \mathscr{F}^s, f \trianglerighteq f^s$ such that $F^{**}(f) \geq F^{**}(f^s) = G^{**}(f^s) > G^{**}(f)$. Thus G^{**} ranks f^s higher than f which contradicts monotonicity. □

Proof of Theorem 3.2. By Theorem 3.1 $F^{**} = F^*$ represents $\succcurlyeq$ on $\mathscr{F}^b$. To verify the representation on $\mathscr{F}$ there are two cases to consider:

(1) $h \succcurlyeq l$ and $F^{**}(l) > F^{**}(h)$:
Consider the following sub-cases:

(i) $h \in \mathscr{F}^b$:
If l is unbounded below, then $F^{**}(l^s) \geq F^{**}(l)$ for some $l^s \in \mathscr{F}^s$. By simple denseness on $\mathscr{F}^b$ there exists $s \in \mathscr{F}^s$ such that $F^*(l^s) \geq F^{**}(l) > F^*(s) > F^*(h)$. As F^{**} represents $\succcurlyeq$ on $\mathscr{F}^b$, $l^s \succ s \succ h \succcurlyeq l$, for all $l^s \trianglerighteq l$. By simple approximation there exists $l^s_* \in \mathscr{F}^s$ such that $s \succ l^s_* \succcurlyeq l$, and the contradiction $l^s_* \succ s \succ l^s_*$ follows.
If l is unbounded above, by the definition of F^{**} there must exist l^s, $l \trianglerighteq l^s$ such that $F^{**}(l) \geq F^*(l^s) > F^*(h)$. By monotonicity, a contradiction $h \succcurlyeq l^s \succ h$ resulted.

(ii) h is unbounded below:
By the definition of F^{**} there must exist $h^s \trianglerighteq h$ such that $F^{**}(l) > F^*(h^s) \geq F^{**}(h)$. By monotonicity $h^s \succcurlyeq h \succcurlyeq l$ which is the case (i) of bounded h.

(iii) h is unbounded above:
If $l \in \mathscr{F}^b$, by the definitions of F^{**}, there must exist h^s such that $F^*(l) > F^{**}(h) \geq F^*(h^s)$. If l is unbounded below, by the definitions of F^{**}, there must exist $l^s, h^s \in \mathscr{F}, l^s \trianglerighteq l, h \trianglerighteq h^s$ such that $F^*(l^s) \geq F^{**}(l) > F^{**}(h) \geq F^*(h^s)$. Thus in both cases by simple denseness on $\mathscr{F}^b$, there exists $s \in \mathscr{F}^s$ such that $F^*(l) > F(s) > F^{**}(h)$ and by case (i) it must be that $l \succ s$. By definition of F^{**} $F^*(s) > F^*(h^s)$ holds for all h^s, $h \trianglerighteq h^s$. As F^{**} represents $\succcurlyeq$ on $\mathscr{F}^b$, $s \succ h^s$. But $h \succcurlyeq l \succ s$, and by simple approximation there must exist $h^s \in \mathscr{F}^s$ such that $h \succcurlyeq h^s \succ s$, a contradiction.

If l is unbounded above, by the definition of F^{**} there must exist l^s, $l \trianglerighteq l^s$ such that $F^{**}(l) \geq F^*(l^s) > F^{**}(h)$. By monotonicity, $h \succcurlyeq l \succcurlyeq l^s$, and we are in the case (i) of bounded l discussed in the previous paragraph.

(2) $h \succ l$ and $F^{**}(h) = F^{**}(l)$:
First, I prove that there exist $h^b, l^b \in \mathscr{F}^b$ such that $h \succcurlyeq h^b \succ l^b \succcurlyeq l$. There are several cases to consider:

(i) h is bounded from both sides.
If l is unbounded from below, by simple approximation there exists l^s such that $h \succ l^s \succcurlyeq l$, and we are done. If l is unbounded above, by non-confoundness there exists $l^b \in \mathscr{F}^b$ such that $h \succ l^b \succcurlyeq l$.

(ii) h is unbounded below.
By non-confoundness there must exist h^b such that $h \succcurlyeq h^b \succ l$. By case (1) and assumption, $F^{**}(h) = F^{**}(h^b) = F^{**}(l)$. Thus we are in the previous case (i) of bounded h.

(iii) h is unbounded above.
By simple approximation there must exist h^s such that $h \succcurlyeq h^s \succ l$. By case (1), $F^{**}(h) = F^{**}(h^s) = F^{**}(l)$ and we are in the previous case (i) of bounded h.

Thus, given $h \succcurlyeq h^b \succ l^b \succcurlyeq l$ we must have, by case (1), $F^{**}(h) \geq F^{**}(h^b) > F^{**}(l^b) \geq F^{**}(l)$ wich contradicts the assumption.

Thus, F^{**} as defined in Eq. (3.2) represents $\succcurlyeq$ on the set of unbounded objects at most from one side. From monotonicity F^{**} must be monotonic with respect to $\trianglerighteq$. This concludes the proof of sufficiency.

To prove necessity, assume that F^{**} is defined as in Eq. (3.2), represents $\succcurlyeq$, and is monotonic with respect to $\trianglerighteq$. Weak ordering and monotonicity are immediate. Simple approximation follow directly from the definitions of

F^* and F^{**}. For non-confoundness, assume that $F^{**}(h) > F^{**}(l)$ and l is unbounded above (case of h unbounded below is analogous). By definition of F^{**} there must exist $l^s \in \mathscr{F}^s$ such that $F^{**}(h) > F^{**}(l) > F^{**}(l^s)$. If h is bounded below then by definition of F^{**} there must exist $h^s, h \trianglerighteq h^s$ such that $F^{**}(h) \geq F^{**}(h^s) > F^{**}(l)$. If h is unbounded below, by definition of F^{**} there must exist $h^s \in \mathscr{F}^s$ such that $F^{**}(h^s) \geq F^{**}(h) > F^{**}(l)$. In both cases of bounded or unbounded h, by simple denseness on $\mathscr{F}^s$ there must exists $s \in \mathscr{F}^s$ such that $F^{**}(h) > F^{**}(s) > F^{**}(l)$, and as F^{**} represents $\succcurlyeq$, non-confoundness holds.

By Theorem 3.1 F^{**} is unique on $\mathscr{F}^b$. To prove uniqueness on $\mathscr{F} \backslash \mathscr{F}^s$ assume that G^{**} also represents $\succcurlyeq$, and $G^{**} = F^{**}$ on $\mathscr{F}^b$, but $G^{**}(f) > F^{**}(f)$ for some $f \in \mathscr{F} \backslash \mathscr{F}^b$ (case $G^{**}(f) < F^{**}(f)$ is analogous). If there exists $b \in \mathscr{F}^b$ such that $b \succcurlyeq f$[25] then $G^{**}(f) > G^{**}(b) = F^{**}(b) \geq F^{**}(f)$. Hence F^{**} and G^{**} cannot both represent $\succcurlyeq$. □

Proof of Theorem 3.3. By assumption of the theorem, $F = F^*$. By the definition of bounded f, F^{**} is well defined for all $f \in \mathscr{F}$ bounded from at least one side (by truncation richness). I prove that F^{**} is also well defined for an unbounded f from both sides, that is, $\inf_\mu\{F^{**}(f_{\vee\mu})\} = \sup_\nu\{F^{**}(f^{\wedge\nu})\}$.

Assume for contradiction $\inf_\mu\{F^{**}(f_{\vee\mu})\} > \sup_\nu\{F^{**}(f^{\wedge\nu})\}$. By the definitions of F^{**} there must exist bounded objects b_+, b_- such that $F^{**}(f_{\vee\mu'}) \geq F^*(b_+) > F^*(b_-) \geq F^{**}(f^{\wedge\nu'})$ for some $\mu', \nu' \in \mathscr{F}^c$. And by simple denseness on $\mathscr{F}^b$, there exists $s \in \mathscr{F}^s$ such that $\inf_\mu\{F^{**}(f_{\vee\mu})\} > F(s) > \sup_\nu\{F^{**}(f^{\wedge\nu})\}$ and $s \not\sim f$. If $f \succ s$ (case $f \prec s$ is similar), by truncation approximation there exists $f^{\wedge\nu}$ such that $f \succcurlyeq f^{\wedge\nu} \succ s$ and $f \trianglerighteq f^{\wedge\nu}$. As $F(s) > F^{**}(f^{\wedge\nu})$, by the definition of $F^{**}(f^{\wedge\nu})$ there must exist $f^{\wedge\nu}_{\vee\mu}$ such that $F(s) > F^*(f^{\wedge\nu}_{\vee\mu})$, and by Theorem 3.1 $f^{\wedge\nu} \succ s \succ f^{\wedge\nu}_{\vee\mu}$, contradicting truncation monotonicity, $f^{\wedge\nu}_{\vee\mu} \succcurlyeq f^{\wedge\nu}$. Thus F^{**} is well defined.

Sufficiency is proven by contradiction. If F^{**} does not represent $\succcurlyeq$, there are two cases to consider:

(1) $h \succcurlyeq l$ and $F^{**}(l) > F^{**}(h)$:
Consider the following sub-cases:

(i) $h \in \mathscr{F}^b$:
If l is unbounded below and bounded above, then $F^{**}(l_{\vee\mu}) \geq F^{**}(l)$ for some μ. By simple denseness on $\mathscr{F}^b$ there exists $s \in \mathscr{F}^s$ such that $F^*(l_{\vee\mu}) \geq F^{**}(l) > F(s) > F(h)$. As F^{**} represents $\succcurlyeq$ on $\mathscr{F}^b$, $l_{\vee\mu} \succ s \succ h \succcurlyeq l$ for all $l_{\vee\mu}$. But by truncation approximation

[25] In case when f is bounded above this holds by construction.

there exists $\mu^* \in \mathscr{C}$ such that $s \succ l_{\vee\mu^*} \succcurlyeq l$, and the contradiction $l_{\vee\mu^*} \succ s \succ l_{\vee\mu^*}$ follows.

If l is unbounded above, by the definition of F^{**} there must exist $l^{\wedge\nu}$ such that $F^{**}(l) \geq F^{**}(l^{\wedge\nu}) > F^{**}(h)$. By truncation monotonicity, $h \succcurlyeq l^{\wedge\nu}$, and we are in the case of bounded above l discussed in the previous paragraph.

(ii) h is bounded above and unbounded below:
By the definition of F^{**} there must exist $h_{\vee\mu}$ such that $F^{**}(l) > F^{*}(h_{\vee\mu}) \geq F^{**}(h)$. By truncation monotonicity $h_{\vee\mu} \succcurlyeq h \succcurlyeq l$ which is the case (i) of bounded form above h.

(iii) h is unbounded above and bounded below:
If $l \in \mathscr{F}^b$, by the definitions of F^{**}, there must exist $h^{\wedge\nu}$ such that $F^{**}(l) > F^{**}(h) \geq F^{**}(h^{\wedge\nu})$. If l is unbounded below, by the definitions of F^{**}, there must exist $l_{\vee\mu}$ and $h^{\wedge\nu}$ such that $F^{**}(l_{\vee\mu}) \geq F^{**}(l) > F^{**}(h) \geq F^{**}(h^{\wedge\nu})$. In both cases, by simple denseness there exists $s \in \mathscr{F}^s$ such that $F^{**}(l) > F(s) > F^{**}(h)$ and by case (i) it must be that $l \succ s$. For all truncations $h^{\wedge\nu}$, $F(s) > F^{**}(h^{\wedge\nu})$, and as F^{**} represents $\succcurlyeq$ on $\mathscr{F}^b$, $s \succ h^{\wedge\nu}$. But $h \succcurlyeq l \succ s$, and by truncation approximation there must exist $\nu^* \in \mathscr{C}$ such that $h \succcurlyeq h^{\wedge\nu^*} \succ s$, a contradiction.

If l is unbounded above, by the definition of F^{**} there must exist $l^{\wedge\nu}$ such that $F^{**}(l) \geq F^{**}(l^{\wedge\nu}) > F^{**}(h)$. By truncation monotonicity, $h \succcurlyeq l^{\wedge\nu}$, and we are in the case of bounded above l discussed in the previous paragraph.

(iv) h is unbounded from both sides:
There must exist $h_{\vee\mu}$ such that $F^{**}(l) > F^{**}(h_{\vee\mu}) \geq F^{**}(h)$. Because $h_{\vee\mu} \succcurlyeq h \succcurlyeq l$ we are in the case (iii) from above.

(2) $h \succ l$ and $F^{**}(h) = F^{**}(l)$:
First, I prove that there exist $h^b, l^b \in \mathscr{F}^b$ such that $h \succcurlyeq h^b \succ l^b \succcurlyeq l$. There are several cases to consider:

(i) h is bounded from both sides.
If l is unbounded below and bounded above, by truncation approximation there exists $l_{\vee\mu}$ such that $h \succ l_{\vee\mu} \succcurlyeq l$, and we are done.

If l is unbounded above, by non-confoundness there exists $l^b \in \mathscr{F}^b$ such that $h \succ l^b \succcurlyeq l$.

(ii) h is bounded from above and unbounded below.
By non-confoundness there must exist h^b such that $h \succcurlyeq h^b \succ l$. By

case (1), $F^{**}(h) = F^{**}(h^b) = F^{**}(l)$. Thus we are in the previous (i) of bounded h.

(iii) h is unbounded above.
By truncation approximation there must exist $h^{\wedge\nu}$ such that $h \succcurlyeq h^{\wedge\nu} \succ l$. By case (1), $F^{**}(h) = F^{**}(h^{\wedge\nu}) = F^{**}(l)$ and we are in one of the previous cases (i) or (ii).

Given $h \succcurlyeq h^b \succ l^b \succcurlyeq l$ we must have by case (1), $F(h)^{**} \geq F^{**}(h^b) > F^{**}(l^b) \geq F^{**}(l)$.

Thus, F^{**} as defined in Eq. (3.3) represents $\succcurlyeq$. From monotonicity and truncation monotonicity, F^{**} must be monotonic with respect to $\trianglerighteq$ and truncation. This concludes the proof of sufficiency.

To prove necessity, assume that F^{**}, as defined in Eq. (3.3), represents $\succcurlyeq$, is monotonic with respect to $\trianglerighteq$ and truncation. Weak ordering, monotonicity and truncation monotonicity follow immediately. Truncation approximation follows directly from the definitions of F^* and F^{**}. For non-confoundness, assume that $F^{**}(h) > F^{**}(l)$ and l is unbounded above (case of h unbounded below is analogous). If h is bounded from both sides, set $h^b = h$. If h is unbounded above then by definition of F^{**} there must exist $h^{\wedge\nu}$ such that $F^{**}(h) \geq F^{**}(h^{\wedge\nu}) > F^{**}(l)$. If $h^{\wedge\nu}$ is bounded from below, set $h^b = h^{\wedge\nu}$, else set $h^b = h^{\wedge\nu}_{\vee\mu}$ for some μ. Hence $F^{**}(h^b) \geq F^{**}(h^{\wedge\nu}) > F^{**}(l)$. If l is bounded below then there exists $l^{\wedge\nu}$ such that $F^{**}(l) \geq F^{**}(l^{\wedge\nu})$. By simple denseness on $\mathscr{F}^b$ there exists s such that $F^{**}(h^{\wedge\nu}) > F^{**}(s) > F^{**}(l)$. As F^{**} represents $\succcurlyeq$, non-confoundness holds. If l is unbounded below, by definition of F^{**} there must exist $l_{\vee\mu}$ such that $F^{**}(h) > F^{**}(l_{\vee\mu}) \geq F^{**}(l)$ and we are in the previous case of bounded below l.

By Theorem 3.1 F^{**} is unique on $\mathscr{F}^b$. To prove uniqueness on $\mathscr{F} \backslash \mathscr{F}^s$ assume that G^{**} also represents $\succcurlyeq$, and $G^{**} = F^{**}$ on $\mathscr{F}^b$, but $G^{**}(f) > F^{**}(f)$ for some $f \in \mathscr{F} \backslash \mathscr{F}^b$ (case $G^{**}(f) < F^{**}(f)$ is analogous). If there exists $b \in \mathscr{F}^b$ such that $b \succcurlyeq f$[26] then $G^{**}(f) > G^{**}(b) = F^{**}(b) \geq F^{**}(f)$. Hence F^{**} and G^{**} cannot both represent $\succcurlyeq$.

□

Proof of Theorem 3.5. By Lemma 3.4, truncation monotonicity holds. By Theorem 3.1 F^* represents $\succcurlyeq$ on $\mathscr{F}^b$. From the definition of F^*, for any $\epsilon \in (F^*(l^b), F^*(h^b))$ there exist $l^s, h^s \in \mathscr{F}^s$ such that $F^*(h^b) \geq F(h^s) > \epsilon > F(l^s) \geq F^*(l^b)$, and by simple-densenes on F^s, simple denseness on $\mathscr{F}^b$ immediately follows. By Theorem 3.3 and Lemma 3.4 the desired representation holds. □

[26] In case when f is bounded above this holds by construction.

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