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Expected utility without continuity: A comment on Delbaen et al. (2011)

Vitalie Spinu, Peter P. Wakker *

Econometric Institute, Erasmus University, P.O. Box 1738, Rotterdam, 3000 DR, The Netherlands

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ABSTRACT

This paper presents preference axiomatizations of expected utility for nonsimple lotteries while avoiding continuity constraints. We use results by Fishburn (1975), Wakker (1993), and Kopylov (2010) to generalize results by Delbaen et al. (2011). We explain the logical relations between these contributions for risk versus uncertainty, and for finite versus countable additivity, indicating what are the most general axiomatizations of expected utility existing today.

1. Introduction

von Neumann and Morgenstern (1947) provided a well-known preference axiomatization of expected utility (EU) for simple lotteries. Delbaen et al. (2011, DDK hereafter) extended it to nonsimple lotteries while avoiding continuity constraints for utility. We present a simplified proof of their results using the more general Theorem 3 of Fishburn (1975). Generalizations by Wakker (1993) to finitely additive lotteries and by Kopylov (2010) to uncertainty are also discussed. Because of these two generalizations, we also cover the domains of de Finetti (1937) and Savage (1954). We show how to apply the uncertainty models of the latter three authors to risk, providing further generalizations there. Our analysis shows the relations between different EU derivations in the literature. In particular, we generalize DDK's result to general (possibly nonreal) outcomes and to more general lottery domains. The latter may (but need not) contain finitely additive rather than countably additive lotteries, and need not be convex.

We assume a *preference relation* $\succcurlyeq$ on a convex set $\mathcal{M}$ of *lotteries* (measurable probability distributions over an arbitrary *outcome* set S, with all degenerate lotteries δ_x included). Then weak ordering, an independence condition, and an Archimedean condition are necessary and sufficient for the existence of an affine functional U that represents $\succcurlyeq$. Defining $u(x) = U(\delta_x)$, the affine functional is uniquely determined as the EU of u on the linear space spanned by the degenerate lotteries, i.e., on the set of *simple* (finite-support) lotteries. Nonsimple lotteries are linearly independent of the degenerate lotteries, and here the affine functional can in principle be chosen freely. It can differ from EU as examples can show.

The extensions of EU to nonsimple lotteries provided in the literature often assumed S to be a subinterval of the reals. Then monotonicity or continuity-in-probability usually is enough to ensure EU for bounded lotteries, because these can be sandwiched between simple lotteries. As pointed out by DDK (p. 401), several authors used a continuity condition known as weak continuity, which in fact is restrictive, implying continuity-in-probability and continuity of utility u.

DDK argued for relaxing the aforementioned continuity constraints because they considered them not to be normative (DDK, p. 401). Another argument against continuity is that its observability status is problematic in the presence of other preference conditions (Krantz et al., 1971, Section 9.1; Pfanzagl, 1968, Section 6.6; Schmeidler, 1971; Suppes, 1974, Section 2). Thus continuity is not as innocuous and "merely technical" as has sometimes been suggested (Arrow, 1971, p. 48; Drèze, 1987, p. 125). This point adds to the interest of DDK's approach.

DDK (Theorem 1.2) showed that, if the domain of preference consists of the set $\mathcal{M}$ of all countably additive lotteries, then the usual stochastic dominance condition is enough to imply EU with no continuity constraint needed for u. Boundedness of u is implied here. This result is appealing because, unlike preceding results in the literature, it does not use advanced concepts and is accessible to a wide audience.

* Corresponding author. Tel.: +31 0 10 408 12 65.
E-mail address: wakker@ese.eur.nl (P.P. Wakker).

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DDK provided two generalizations of their Theorem 1.2. Their Proposition 2.2 deals with lotteries with compact support, and their Theorem 2.5 deals with lotteries that have a finite ψ expectation for some continuous function $\psi : S \to [1, \infty)$. Both results generalize the case of bounded u, allowing for unbounded u.

This note focuses on DDK's derivations of EU. DDK also provided continuity results with respect to some metric topologies, to which we have nothing to add. We do not discuss these results further. For brevity, we follow DDK in assuming an affine functional U[1] throughout and omitting the aforementioned necessary and sufficient preference conditions for the existence of U. For the preference conditions that we introduce later to be directly observable, they should not use U or u as input, and none of the preference conditions in this paper will do so.

2. Deriving the results of DKK from Fishburn's (1975) Theorem 3

Fishburn's (1975) Sections 1–4 consider a general setup where the preference domain $\mathcal{M}$ is allowed to contain finitely additive lotteries. We reproduce Theorem 3 from Fishburn's Section 5. As did DDK, this theorem focuses on countably additive lotteries and does not allow finitely additive lotteries in the preference domain. In Fishburn's analysis, the outcome set S is allowed to be general. Denote by $\mu^{\wedge x}$ the conditional distribution of μ over outcomes weakly less preferred than x, i.e., over $u^{-1}(-\infty, u(x)]$, whenever the latter set is μ nonnull. Similarly, by $\mu_{\vee x}$ we denote the conditional distribution of μ over outcomes weakly preferred to x, i.e., over $u^{-1}[u(x), +\infty)$, again for outcomes x where the latter set is μ nonnull. When using this notation we implicitly assume that the distributions are well defined, i.e., the conditioning events are nonnull. A *preference interval* contains, for every pair of elements x, z, all outcomes y with $x \succcurlyeq y \succcurlyeq z$. That is, a preference interval is the u inverse of an interval. Fishburn made the following assumptions about the preference domain.

Axiom 1 (*0: Fishburn, 1975; A0.2: Fishburn, 1982, Section 3.3*). [Structural Assumption] $\mathcal{M}$ is a set of countably additive lotteries over a general set S, measurable with respect to an algebra on S that contains all singletons and every preference interval. $\mathcal{M}$ contains all degenerate lotteries and is convex. There exists an affine functional U on $\mathcal{M}$ that represents a preference relation $\succcurlyeq$ on $\mathcal{M}$. $\mathcal{M}$ is conditionally closed, i.e., if $f \in \mathcal{M}$ then the lottery conditional on a nonnull preference interval $A \subset S$, *denoted* f_A, is also in $\mathcal{M}$.

Fishburn used the following two conditions. The first one is stochastic dominance imposed only if at least one lottery involved is degenerate.

Axiom 2 (4′*: Fishburn, 1975;* A4**: Fishburn, 1982, Section 3.4*). If $\mu \in \mathcal{M}, A \subseteq S,\ \mu(A) = 1$, and $y \in S$, then $\mu \succcurlyeq y$ if $x \succcurlyeq y$ for all $x \in A$. Similarly, $y \succcurlyeq \mu$ if $y \succcurlyeq x$ for all $x \in A$.

The second condition restricts the set of unbounded lotteries and is the main one to imply that all lotteries have a finite EU, equal to their U value.

Axiom 3 (5′*: Fishburn, 1975;* A5**: Fishburn, 1982, Section 3.4*). If $\mu, \nu \in \mathcal{M}$, ν is simple, and $\mu \succ \nu$ then $\mu^{\wedge x} \succcurlyeq \nu$ for some $x \in S$. If $\mu, \nu \in \mathcal{M}$, ν is simple, and $\mu \prec \nu$ then $\mu_{\vee y} \preccurlyeq \nu$ for some $y \in S$.

We now reproduce Fishburn's EU derivation.

[1] This by definition means that U is real valued and does not take the values ∞ or $-\infty$.

Theorem 1 (*3: Fishburn, 1975; 3.4: Fishburn, 1982*). *Assume that Axiom 1 holds. Then the affine U is an EU functional if and only if Axioms 2 and 3 hold.*

We next show that the conditions of Fishburn's Theorem 3 hold under the assumptions of DDK's EU results. Fishburn's Axiom 0 holds in all these results. Conditional closedness, trivially satisfied in DDK's Theorem 1.2 and Proposition 2.2, also holds in their Theorem 2.5 because, if the Lebesgue integral of ψ is defined over the space S, it surely is defined over subsets of S. We next derive Axioms 2 and 3, i.e., Fishburn's (1975) Axioms 4′ and 5′.

Lemma 2. *Assume that S is an interval, and (weak) stochastic dominance. If $\mathcal{M}$ contains all countably additive lotteries (DDK Theorem* 1.2*), or all countably additive lotteries with compact support (DDK Proposition* 2.2*), or all countably additive lotteries that have a finite first moment with respect to a continuous function $\psi : S \to [1, \infty)$ (DDK's Theorem* 2.5*), then Axioms 2 and 3 hold.*

Corollary 3. *The EU derivations in DDK (their Theorem* 1.2*, Proposition* 2.2*, and Theorem* 2.5*) follow from Theorem* 3 *in Fishburn (1975).*

3. Related work

Wakker (1993, Theorem 3.6) provided a generalization of Fishburn's (1975, 1982) EU representations. As in DDK, he did not require continuity of utility. And as in Fishburn's Sections 1–4, Wakker considered cases where $\mathcal{M}$ is allowed (but not required) to contain finitely additive probability measures. Wakker also considered general outcome sets. Instead of stochastic dominance (properly extended to general outcomes), he used a conditional monotonicity condition (a counterpart of Savage's P7) which, under the other assumptions, is equivalent to stochastic dominance under countable additivity, but is stronger under finite additivity. Wakker's Example 4.10 showed that stochastic dominance is too weak under finite additivity. His domain assumptions and truncation condition were more general than those of Fishburn (1975, 1982). They only require the availability of all simple functions, the existence of an equivalent simple lottery for each lottery, and the existence of truncations. This domain need not be convex. Then necessary and sufficient results were given.

Kopylov (2010) provided a counterpart to the aforementioned works for Savage's (1954) EU representation. Kopylov and Savage did not consider decision under risk with lotteries, but decision under uncertainty, with a state space T and acts mapping states to outcomes. However, decision under risk can be considered to be a special case of decision under uncertainty (Kothiyal et al., 2011, Appendix C). To see this point in the present context, assume the DDK model, with an interval outcome set. Define $T = [0, 1]$ as state space with the usual Borel sigma algebra. We endow T with the uniform probability distribution. Preferences between acts are determined by preferences in the DDK model between the lotteries that the acts generate over the outcomes. In this way, DDK's models become models of Kopylov. Then all the conditions in Kopylov's Theorem 1 are satisfied in the results of DDK, and DDK's expected utility follows from the expected utility that Kopylov's result gives. In this way, Kopylov's theorem is also more general.

Kopylov (2010) also contains remarkable results for intertemporal choice. His Corollary 4 provides the first axiomatization in the literature of constant discounting in integral form. It is remarkable that this form, one of the most widely used evaluation formulas in the literature, had not received a preference axiomatization before. Axiomatizations of constant discounting had as yet been provided exclusively for discrete summations rather than continuous integrations over time (Koopmans, 1972; Bleichrodt et al., 2008).

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Appendix. Proofs

Proof of Lemma 2. DDK's Theorem 1.2 is a corollary of their Theorem 2.5 (take ψ constant). Cases of compact supports, containing their maximum α and minimum, are simple (see the first lines of the two cases below). Hence we focus on DDK's Theorem 2.5. The following proof will amount to constructing, for contradiction, St. Petersburg lotteries in the domain of DDK's Theorem 2.5.

Part 1 (*Proof of Axiom 3*)**.** Take a lottery μ with support A with supremum $\alpha := \inf\{\beta : \mu(\beta, \infty) = 0\}$. Assume, for contradiction, $\mu \succ \nu$ but $\mu^{\wedge x} \preccurlyeq \nu$ for every x for which $\mu^{\wedge x}$ is well defined. (Cases of $\mu \prec \nu$ and always $\mu_{\vee x} \succcurlyeq \nu$ similarly lead to a contradiction.) $\alpha \notin A$ because otherwise a contradiction $\mu^{\wedge a} = \mu \succ \nu \succcurlyeq \mu^{\wedge a}$ would follow. Take a sequence of outcomes $a_j \in A$ strictly increasing to α. $\lambda_j := \mu(a_j, \infty) > 0$ tends to 0 by countable additivity. For j large enough, $\lambda_j < 1$. We assume this for all j (take a subsequence and reindex). Now $\mu^{\wedge a_j}$ is well defined.

Decompose

$$\mu = \lambda_j \mu_j^+ + (1 - \lambda_j)\mu_j^- \tag{1}$$

with $\mu_j^+ = \mu_{(a_j,\infty)}$ and $\mu_j^- = \mu_{(-\infty,a_j]}$. Both conditional distributions are well defined because $0 < \lambda_j < 1$, and are contained in the preference domain (finite ψ expectation as with μ). $U(\mu_j^-) \leq U(\nu)$ because $\nu \succcurlyeq \mu^{\wedge a_j} \succcurlyeq \mu_j^-$ ($\mu^{\wedge a_j}$ may involve outcomes larger than but equivalent to a_j).

$U(\mu) - U(\mu_j^-) \geq (U(\mu) - U(\nu)) =: 2\Delta > 0$. Substituting Eq. (1) for $U(\mu)$, we get

$$\lambda_j U(\mu_j^+) - \lambda_j U(\mu_j^-) \geq 2\Delta. \tag{2}$$

By stochastic dominance, $U(\mu_j^-)$ is nondecreasing in j; it is bounded above by $U(\nu)$.

Observation. With $\mu, A, \alpha, \alpha_j, \lambda_j, \Delta$, and Eqs. (1) and (2) as above, a contradiction results.

$\lambda_j U(\mu_j^-)$ tends to 0 as does λ_j. There must exist J such that

$$\lambda_j U(\mu_j^+) \geq \Delta \tag{3}$$

for all $j \geq J$. We may assume that $J = 1$ (take a subsequence and reindex). By countable additivity, $\int_{(a_j,\infty)} \psi d\mu$ tends to 0. By taking an increasing subsequence and reindexing, we can get

$$\int_{(-\infty,\infty)} \psi \lambda_j d\mu_j^+ = \int_{(a_j,\infty)} \psi d\mu \leq \frac{1}{2^j} \tag{4}$$

for all j. Given $\psi \geq 1$, also $\lambda_j \leq 1/2^j$ and $\mu_n^* = \sum_{j=1}^n \lambda_j \mu_j^+ + (1 - \sum_{j=1}^n \lambda_j) a_1$ is well defined. Obviously, given positivity of ψ, $\int_{-\infty}^{+\infty} \psi d\mu_n^* < 1 + \psi(a_1)$, which implies that μ_n^* is in the domain of U for all n, also for $n = \infty$.

By stochastic dominance, $U(\mu_\infty^*) \geq U(\mu_n^*)$ for all n. Hence, by Eq. (3), $U(\mu_\infty^*) \geq \sum_{j=1}^n \Delta + (1 - \sum_{j=1}^n \lambda_j) U(a_1)$ for all n, and it cannot be finite. A contradiction has resulted.

Part 2 (*Proof of Axiom 2*)**.** Axiom 2 trivially follows by restricting stochastic dominance to the case where one of the two lotteries involved is degenerate if stochastic dominance is strict, or if stochastic dominance is related to the preference order over outcomes rather than to the natural order on $\mathbb{R}$. For the weak stochastic dominance with respect to the natural order on $\mathbb{R}$ used by DKK, the derivation is nontrivial, as can be inferred from their Example 2.1.[2]

Assume, for contradiction, $\mu \succ x$ but $y \preccurlyeq x$ for every y in the support A of μ. (The case of $\mu \prec x$ but $y \succcurlyeq x$ is similar.) Take α, α_j, and λ_j as in Part 1. Again $\alpha \notin A$ (otherwise $\alpha \succcurlyeq \mu \succ x \succcurlyeq \alpha$ gives a contradiction). Again we have $0 < \lambda_j < 1$, and we get Eq. (1). By stochastic dominance, $U(\mu_j^-) \leq U(a_j) \leq u(x)$. $U(\mu_j^-)$ is nondecreasing in j and bounded above by $u(x)$. $U(\mu) - U(\mu_j^-) \geq U(\mu) - U(x) =: 2\Delta > 0$, and we get Eq. (2). From here on we proceed as after the Observation in Part 1.

References

Arrow, Kenneth J., 1971. Essays in the Theory of Risk-Bearing. North-Holland, Amsterdam.

Bleichrodt, Han, Rohde, Kirsten I.M., Wakker, Peter P., 2008. Koopmans' constant discounting for intertemporal choice: a simplification and a generalization. Journal of Mathematical Psychology 52, 341–347.

de Finetti, Bruno, 1937. La Prévision: ses lois logiques, ses sources subjectives. Annales de l'Institut Henri Poincaré 7, 1–68;
Translated into English by Kyburg, Henry E., 1964. Foresight: its logical laws, its subjective sources. In: Kyburg, Henry E., Smokler, Howard E. (Eds.), Studies in Subjective Probability. Wiley, New York, pp. 53–118. 1980. second ed. Krieger, New York.

Delbaen, Freddy, Drapeau, Samuel, Kupper, Michael, 2011. A von Neumann–Morgenstern representation result without weak continuity assumption. Journal of Mathematical Economics 47, 401–408.

Drèze, Jacques H., 1987. Essays on Economic Decision Under Uncertainty. Cambridge University Press, London.

Fishburn, Peter C., 1975. Unbounded expected utility. Annals of Statistics 3, 884–896.

Fishburn, Peter C., 1982. The Foundations of Expected Utility. Reidel, Dordrecht.

Koopmans, Tjalling C., 1972. Representations of preference orderings with independent components of consumption, & representations of preference orderings over time. In: McGuire, C.B., Radner, Roy (Eds.), Decision and Organization. North-Holland, Amsterdam, pp. 57–100.

Kopylov, Igor, 2010. Simple axioms for countably additive subjective probability. Journal of Mathematical Economics 46, 867–876.

Kothiyal, Amit, Spinu, Vitalie, Wakker, Peter P., 2011. Prospect theory for continuous distributions: a preference foundation. Journal of Risk and Uncertainty 42, 195–210.

Krantz, David H., Luce, R.Duncan, Suppes, Patrick, Tversky, Amos, 1971. Foundations of Measurement, Vol. I (Additive and Polynomial Representations). Academic Press, New York.

Pfanzagl, Johann, 1968. Theory of Measurement. Physica-Verlag, Vienna.

Savage, Leonard J., 1954. The Foundations of Statistics. Wiley, New York, (1972. second ed. Dover Publications, New York).

Schmeidler, David, 1971. A condition for the completeness of partial preference relations. Econometrica 39, 403–404.

Suppes, Patrick, 1974. The measurement of belief. Journal of the Royal Statistical Society 36, 160–191.

von Neumann, John, Morgenstern, Oskar, 1947. Theory of Games and Economic Behavior. Princeton University Press, Princeton, NJ.

Wakker, Peter P., 1993. Unbounded utility for Savage's foundations of statistics, and other models. Mathematics of Operations Research 18, 446–485.

[2] We missed this point in an earlier version of our paper. Thanks to Samuel Drapeau for pointing this out to us. Note here that DDK's term sure-thing principle refers simply to Fishburn's Axiom 5′, i.e. our Axiom 2, and not to what is called the sure-thing principle in most of the literature.