Expected Utility without Continuity: A Comment on Delbaen, Drapeau, and Kupper (2011)

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Abstract

This paper considers preference foundations of expected utility that do not assume continuous utility. Using results by Fishburn, we generalize results by Delbaen, Drapeau, and Kupper (2011).

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1 Introduction

One of the most famous preference foundations provided in the literature is the one of expected utility (EU) by von Neumann and Morgenstern (1947, 1953). A *preference relation* $\succcurlyeq$ is given on a convex set $\mathcal{M}$ of *lotteries* (probability distributions over an arbitrary *outcome* set S, with all degenerate lotteries δ_x included). Then weak ordering, an independence condition, and an Archimedean condition are necessary and sufficient for the existence of an affine functional U that represents $\succcurlyeq$. Defining $u(x) = U(\delta_x)$, the affine functional is uniquely determined as the EU of u on the linear space spanned by the degenerate lotteries, i.e., on the set of *simple* (finite-support) lotteries. Non-simple lotteries are linearly independent of the degenerate lotteries, and here the affine functional is in principle free to choose. It can differ from EU as examples can show.

A number of papers provided extra conditions to extend EU to non-simple lotteries. Those papers often considered the case where S is a subinterval of the reals. Then a monotonicity or continuity-in-probability condition usually is enough to ensure EU for bounded lotteries, because these can be sandwiched between simple lotteries. As pointed out by Delbaen, Drapeau, & Kupper (2011 p. 401; DDK hereafter), several authors used a continuity condition known as weak continuity, which in fact is restrictive and implies that utility u is continuous. Subsequent extensions to nonbounded prospects are usually more complex.

DDK argued for the interest of relaxing the aforementioned continuity conditions, which these authors considered not to be normative. Although it has sometimes been suggested that continuity is an innocuous assumption (Arrow 1971 p. 48; Drèze 1987 p. 125), several authors have explained its problematic empirical status (Krantz et al. 1971 Section 9.1; Pfanzagl 1968 Section 6.6; Schmeidler 1971; Suppes 1974 Section 2), adding to the interest of DDK's approach.

DDK (Theorem 1.2) showed that if the domain of preference consists of the set $\mathcal{M}$ of all countably additive Lebesgue measurable lotteries, then the usual stochastic dominance condition is enough to imply EU with no continuity restriction needed for u. Boundedness of u is implied here. Their theorem is especially appealing because, unlike preceding results in the literature, it does not use advanced concepts and is accessible to a wide audience.

DDK provided two generalizations of their Theorem 1.2. Their Proposition 2.2 restricts the domain of preference to lotteries with compact support, and their Theorem 2.5 restricts the domain to lotteries that have a finite ψ expectation for some function $\psi : S \to [1, \infty)$. Both results are generalizations because they allow u to be unbounded.

This note discusses the relationship between DDK's results and some preceding results in the literature that did not impose continuity of utility either. We will focus on DDK's derivations of EU. DDK also provided a number of continuity results with respect to some metric topologies, to which we have nothing to add, and we do not discuss these results further. In particular, we will provide a simple proof of DDK's EU derivations using the more general Theorem 3 of Fishburn (1975; reproduced in Fishburn 1982 Theorem 3.4). We describe generalizations to general (possibly nonreal) outcomes, to more general lottery domains, and to finitely additive lotteries. For brevity, we follow DDK in assuming an affine functional U throughout and omitting the aforementioned necessary and sufficient preference conditions for U.

2 Deriving the results of DKK from Fishburn's (1975) Theorem 3

Fishburn's (1975) Sections 1-4 consider a general setup where the preference domain $\mathcal{M}$ is allowed to contain finitely additive lotteries. We reproduce Theorem 3 from Fishburn's Section 5. As did DDK, this theorem focuses on countably additive lotteries and does not allow finitely additive lotteries in the preference domain. In Fishburn's analysis, the outcome set S is allowed to be general. Denote by $\mu^{\wedge x}$ the conditional distribution of μ over outcomes weakly less preferred than x, i.e. over $u^{-1}(-\infty, u(x)]$, whenever the latter set is μ nonnull. Similarly, by $\mu_{\vee x}$ we denote the conditional distribution of μ over $u^{-1}[u(x), +\infty)$, again for outcomes x where the latter set is μ nonnull. When using this notation we implicitly assume that the distributions are well defined, with the conditioning events nonnull. A *preference interval* is the u inverse of an interval. Fishburn made the following assumptions about the preference domain.

Axiom 1 (0: Fishburn 1975; A0.2: Fishburn 1982 Section 3.3) [Structural assumption] *$\mathcal{M}$ is a set of countably additive lotteries over a general set S, measurable with respect to an algebra on S that contains all singletons and every preference interval. $\mathcal{M}$ contains all degenerate lotteries and is convex. An affine functional U is given on $\mathcal{M}$. $\mathcal{M}$ is conditionally closed, i.e. if $f \in \mathcal{M}$ then the lottery conditional on a nonnull preference interval $A \subset S$, f_A, is also in $\mathcal{M}$.*

Fishburn used the following conditions.

Axiom 2 (4': Fishburn 1975; A4*: Fishburn 1982 Section 3.4) *If $\mu \in \mathcal{M}$, $A \subseteq S$, $\mu(A) = 1$, and $y \in S$, then $\mu \succcurlyeq y$ if $x \succcurlyeq y$ for all $x \in A$. Similarly, $y \succcurlyeq \mu$ if $y \succcurlyeq x$ for all $x \in A$.*

Axiom 3 (5': Fishburn 1975; A5*: Fishburn 1982 Section 3.4) *If* $\mu, \nu \in \mathcal{M}$*,* ν *is simple, and* $\mu \succ \nu$ *then* $\mu^{\wedge x} \succcurlyeq \nu$ *for some* $x \in S$*. If* $\mu, \nu \in \mathcal{M}$*,* ν *is simple, and* $\mu \prec \nu$ *then* $\mu_{\vee y} \preccurlyeq \nu$ *for some* $y \in S$*.*

The last condition restricts the set of unbounded lotteries and is the main one to imply that all lotteries have a finite EU, equal to their U value. We now reproduce Fishburn's EU derivation (with the affine functional U assumed).

THEOREM 1 (3: Fishburn 1975; 3.4: Fishburn 1982) *Assume that Axiom 0 holds. Then the affine* U *is an EU functional if and only if Axioms 4' and 5' hold.*

We next show that the conditions of Fishburn's Theorem 3 hold under the assumptions of DDK's EU results. Fishburn's Axiom 0 holds in all these results. Conditional closedness, trivially satisfied in DDK's Theorem 1.2 and Proposition 2.2, also holds in their Theorem 2.5 because if the Lebesgue integral of ψ is defined over the space S, it surely is over subsets of S. We next derive Fishburn's Axioms 4' and 5'.

LEMMA 2 *If* $\mathcal{M}$ *contains all countably additive lotteries (DDK Theorem 1.2), or all countable additive lotteries with compact support (DDK Proposition 2.2), or all countably additive lotteries that have a finite first moment with respect to a function* $\psi : S \to [1, \infty)$ *(DDK's Theorem 2.5), then Axioms 2 and 3 hold.*

COROLLARY 3 *The EU derivations in DDK (their Theorem 1.2, Proposition 2.2, and Theorem 2.5) follow from Theorem 3 in Fishburn (1975).*

3 Related Work

Wakker (1993, Theorem 3.6) provided a generalization of Fishburn's (1975, 1982) EU representations. As with DDK, he did not require continuity of utility. And as in Fishburn's Sections 1-4, Wakker considered cases where $\mathcal{M}$ is allowed (but not required) to contain finitely additive probability measures. Wakker also considered general outcome sets. Instead of stochastic dominance, he used a conditional monotonicity condition (a counterpart of Savage's P7) which, under the other assumptions, is equivalent to stochastic dominance under countable additivity, but is stronger under finite additivity. Wakker's Example 4.10 showed that stochastic dominance is too weak under finite additivity. His domain assumptions and truncation condition were more general than those of Fishburn (1975, 1982). They only require the availability of all simple functions, the existence of an equivalent simple lottery for each lottery, and the existence of truncations. Then necessary and sufficient results were given.

Kopylov (2010) provided a counterpart to the aforementioned works for Savage's (1954) EU representation. Kopylov and Savage did not consider decision under risk with lotteries, but decision under uncertainty, with a state space T and acts that map states to outcomes. However, decision under risk can be considered to be a special case of decision under uncertainty (Kothiyal et al. 2011, Appendix C). For an interval outcome set as in DDK, we can take $T = [0, 1]$ as state space with the usual Lebesgue sigma algebra. We endow T with the uniform probability distribution. Preferences between acts are determined by preferences between the lotteries that they generate over the outcomes. In this way, DDK's models become models of Kopylov. Then all the conditions in Kopylov's Theorem 1 are satisfied in the results of DDK, and in this sense Kopylov's theorem is also more general.

Kopylov (2010) also contains remarkable results for intertemporal choice. His Corollary 4 provides the first axiomatization in the literature of constant discounting in integral form. It is remarkable that this form, one of the most widely used evaluation formulas in the literature, had not received a preference axiomatization before. Axiomatizations of constant discounting had as yet been provided exclusively for discrete constant discounting (Koopmans 1972; Bleichrodt, Rohde, & Wakker 2008).

Appendix. Proofs

PROOF OF LEMMA 2. DDK's Theorem 1.2 is a corollary of their Theorem 2.5 (take ψ constant). Cases of compact supports, containing their maximum α and minimum, are simple (see the first lines of the two cases below). Hence we focus on DDK's Theorem 2.5. The following proof will amount to constructing, for contradiction, St. Petersburg lotteries in the domain of DDK's Theorem 2.5.

Part 1. Proof of Axiom 3. Take a lottery μ with support A with supremeum $\alpha := \inf\{\beta : \mu(\beta, \infty) = 0\}$. Assume, for contradiction, $\mu \succ \nu$ but $\mu^{\wedge x} \preccurlyeq \nu$ for every x for which $\mu^{\wedge x}$ is well defined. (Cases of $\mu \prec \nu$ and always $\mu_{\vee x} \succcurlyeq \nu$ similarly lead to a contradiction.) $\alpha \notin A$ because otherwise a contradiction $\mu^{\wedge a} = \mu \succ \nu \succcurlyeq \mu^{\wedge a}$ would follow. Take a sequence of outcomes $a_j \in A$ strictly increasing to α. $\lambda_j := \mu(a_j, \infty) > 0$ tends to 0 by countable additivity. For j large enough, $\lambda_j < 1$. We assume this for all j (take a subsequence and re-index). Now $\mu^{\wedge a_j}$ is well defined.

Decompose

$$\mu = \lambda_j \mu_j^+ + (1 - \lambda_j)\mu_j^- \tag{1}$$

with $\mu_j^+ = \mu_{(a_j,\infty)}$ and $\mu_j^- = \mu_{(-\infty,a_j]}$. Both conditional distributions are well defined because $0 < \lambda_j < 1$, and are contained in the preference domain (finite ψ expectation as with μ). $U(\mu_j^-) \leq U(\nu)$ because $\nu \succcurlyeq \mu^{\wedge a_j} \succcurlyeq \mu_j^-$ ($\mu^{\wedge a_j}$ may involve outcomes larger than but equivalent to a_j).

$U(\mu) - U(\mu_j^-) \geq (U(\mu) - U(\nu)) =: 2\Delta > 0$. Substituting Eq. 1 for $U(\mu)$, we get

$$\lambda_j U(\mu_j^+) - \lambda_j U(\mu_j^-) \geq 2\Delta. \tag{2}$$

By stochastic dominance, $U(\mu_j^-)$ is nondecreasing in j; it is bounded above by $U(\nu)$.

Observation. With μ, A, α, α_j, λ_j, Δ, and Eqs. 1 and 2 as above, a contradiction results.

$\lambda_j U(\mu_j^-)$ tends to 0 as does λ_j. There must exist J such that

$$\lambda_j U(\mu_j^+) \geq \Delta \tag{3}$$

for all $j \geq J$. We may assume that $J = 1$ (take subsequence and re-index). By countable additivity, $\int_{(a_j,\infty)} \psi d\mu$ tends to 0. By taking an increasing subsequence and reindexing, we can get

$$\int_{(a_j,\infty)} \psi d\mu \leq \frac{1}{2^j} \tag{4}$$

for all j. Given $\psi \geq 1$, also $\lambda_j \leq 1/2^j$ and $\mu_n^* = \sum_{j=1}^n \lambda_j \mu_j^+ + (1 - \sum_{j=1}^n \lambda_j) a_1$ is well defined. Obviously, given positivity of ψ, $\int_{-\infty}^{+\infty} \psi d\mu_n^* < 1 + \psi(a_1)$ which implies that μ_n^* is in the domain of U for all n, also for $n = \infty$.

By stochastic dominance, $U(\mu_\infty^*) \geq U(\mu_n^*)$ for all n. Hence, by Eq. 3, $U(\mu_\infty^*) \geq \sum_{j=1}^n \Delta + (1 - \sum_{j=1}^n \lambda_j) U(a_1)$ for all n, and it cannot be finite. A contradiction has resulted.

Part 2. Proof of Axiom 2. Assume, for contradiction, $\mu \succ x$ but $y \preccurlyeq x$ for every y in the support A of μ. (The case of $\mu \prec x$ but $y \succcurlyeq x$ is similar.) Take α, α_j, and λ_j as in Part 1. Again $\alpha \notin A$ (otherwise $\alpha \succcurlyeq \mu \succ x \succcurlyeq \alpha$ gives a contradiction). Again we have $0 < \lambda_j < 1$, and we get Eq. 1. By stochastic dominance, $U(\mu_j^-) \leq U(a_j) \leq u(x)$.

$U(\mu) - U(\mu_j^-) \geq U(\mu) - U(x) =: 2\Delta > 0$, and we get Eq. 2. From here on we proceed as following the Observation in Part 1.

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